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Entropy and dimension of a chaotic attractor depending on the control parameter

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Abstract. The dependence of the entropy and dimension of the chaotic attractor on the control parameter is investigated by the numerical experiment. Calculations are carried out for one of the simplest systems described by nonlinear equations of dynamics—a rotator driven by an external periodic field. Here, regular and chaotic solutions alternate when the control parameter changes. The numerical experiment shows that the dimension of the chaotic attractor and, as a consequence, its entropy change significantly when the control parameter is in the ranges where, due to intermittency, the transition from chaotic motion to regular motion occurs.

Keywords: nonlinear dynamics, strange attractor, chaotic attractor, probability density, chaos, intermittency, rotator

Introduction

In systems described by equations of nonlinear dynamics, in the presence of dissipation, both regular and chaotic movements are possible (Gonchenko et al. 2017; Schuster, Just 2005). In the case of regular movements, the trajectory of the system in phase space is a closed line, or degenerates into a point. Such sets of points are attractors. In the case of chaotic motion, the trajectory of motion is an infinite non-intersecting line. At the same time, the region in the phase space that restricts the movement of the system is constantly narrowing over time, which means that the phase trajectory also tends to a certain attractor, which is called a chaotic or strange attractor. In this case, the state of the system can be described in terms of probability. It means that one can define a probability density that determines the probability of finding the system in some given region of space, similarly to the probability density in systems described by statistical physics or quantum mechanics (Kuznetsov 2006; Liapzev 2019).

A distinctive feature of the chaotic attractor is that it irregularly fills the region of phase space. The correct definition of the chaotic attractor shows that it has a factional dimension and it is smaller than the dimension of the phase space (Malinetsky, Potapov 2000). A salient aspect of the nonlinear dynamic systems in question is the dependence of the character of motion on some parameters of the problem, which are called control parameters. In particular, with an adiabatic change of any of these parameters, the system can jump from chaotic to regular motion and vice versa. Since in the case of regular motion the attractor is a closed line, its dimension is equal to one. This means that, generally speaking, a change in the control parameter leads to a change in the dimension of the attractor.

Since the measure of chaos is entropy, it is possible to correctly define the concept of entropy for a system whose motion is described by a chaotic attractor (see, for example, (Malinetsky, Potapov 2000)). At the same time, it is natural to assume that the regular movement of the system should manifest itself, at least entropy. It follows from the above that the entropy of the system described by the chaotic attractor should also depend on the control parameters.

It is unlikely that analytical research methods alone are sufficient for the general study of the dependence of the dimension and entropy of chaotic attractors. This makes a numerical experiment useful, at least, for the simplest dynamical systems. One of such systems is a rotator (a system with one-dimensional rotational motion), driven by an external periodic field. The control parameter for such a system is the amplitude of the external field. The purpose of this article is to use numerical methods to investigate the dependence of the entropy and dimension of a chaotic attractor on the control parameters in such systems.

Problem statement. Equation of motion

Consider an electric dipole that can make a one-dimensional rotation (Fig. 1). Such a real system consists of two oppositely charged balls of the same mass connected by a non-conducting rod fixed on a hinge.

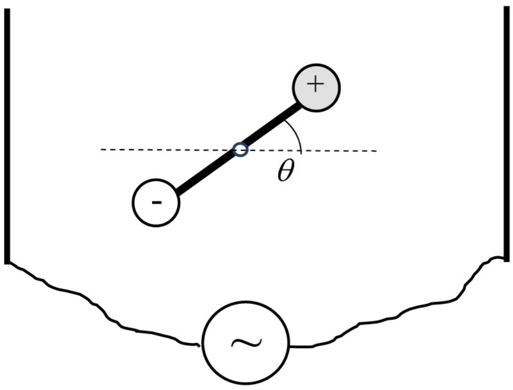

Fig. 1. Rotator scheme

The effect of an external harmonic field can be taken into account by placing such a system between the plates of the capacitor to which the alternating voltage is applied. Note that a similar model can be used to describe the rotational motion of a molecule in a microwave field, since the rotational motion in a good approximation can be described in the framework of a quasi-classical approximation.

We also assume that the rotation is decelerated with the viscous friction force, which is proportional to the angular velocity of rotation. Then the dynamic equation describing the motion has the form:

$$I\ddot{\theta} = qE_0 \sin\theta \cos(\omega t) - \lambda\dot{\theta},$$

where I is the moment of inertia of the dipole, q is the absolute value of charge of the balls, E_0 and ω is the intensity and frequency of the external field, λ is the coefficient of proportionality between the moment of friction and the angular velocity. Here and further the point denotes the derivative of the variable t. To simplify the equation, we perform a large-scale transformation of the time variable. $t' = \omega t$. Then the equation of motion takes the form:

$$\ddot{\theta} + \gamma\dot{\theta} = f\theta \cos t \tag{1}$$

where $\gamma = \dfrac{\lambda}{I\omega}$, $f = \dfrac{qE_0}{I\omega^2}$ (for simplicity, in what follows t will be used instead of t'). Note that this equation and the corresponding Poincare cross section pattern were considered in the monograph (Sagdeev et al. 1988), and the formulation of similar quantum mechanical problem in the absence of dissipation is considered in (Stockmann 1999), where, in particular, it was noted that "systems with harmonic dependence on time are not too popular among theorists".

The differential equation of the second order (1) is reduced to an autonomous system of three differential equations of the first order:

$$\begin{gathered}\dot{\theta} = p,\\ \dot{p} = f \sin\theta \cos\tau - \gamma p.\\ \dot{\tau} = 1.\end{gathered} \tag{2}$$

The variables θ, p and τ form a three-dimensional phase space. This is the minimum value of dimension for which the trajectory can tend to a chaotic attractor, similar to the Lorentz attractor (see, for example, (Grinchenko et al. 2007)). At the same time, depending on the control parameter f, the solution can be both regular and chaotic. Parameter γ characterizing the dissipation, is usually assumed to be small. However, namely its difference from zero determines the tendency of the trajectory in the phase space to the attractor. Unlike the time variable, the variable τ can be considered as periodic with a period of 2π. Taking into account the periodicity of the variable θ, it is convenient to consider the trajectory in the phase space as a line "wound" on the torus (Fig. 2).

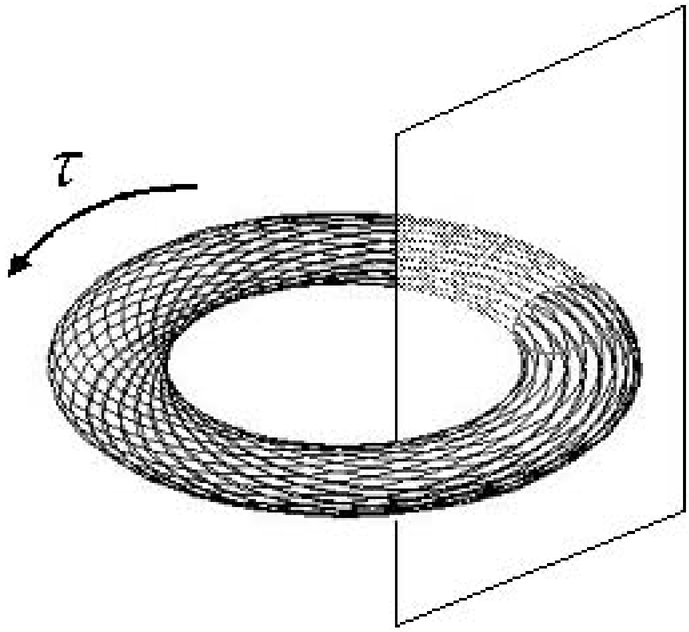

Fig. 2. The trajectory of motion in phase space

The cross section of such a torus by a plane is a Poincare cross section; the corresponding set of points is a fractal.

The control parameter in equation (1), or in the system of equations (2) is the parameter f. Depending on the values of this parameter, the solution can be regular, in which the trajectory of motion in phase space is a closed line taking into account the periodicity of the variables θ and τ, or chaotic, in which the trajectory is an infinite open line. In the first case, the attractor is a limit cycle, and in the second, a chaotic ("strange" in Lorentz's terminology) attractor.

Probabilistic approach

In the case of chaotic motion, the state of the system can be described in the language of probability, for which it is possible to introduce the concept of probability density (Liapzev 2019). Let us define a density probability distribution ρ (τ, θ, p) as follows: for a given value τ, the value $\Delta w = \rho\ (\tau, \theta, p)\ \Delta\tau\ \Delta\theta\ \Delta p$ is equal to the probability that the trajectory of the system passes in the region [τ, τ = $\Delta\tau$; θ, θ = $\Delta\theta$:p, $p + \Delta p$] and the normalization condition is set:

$$\int_0^{2\pi} d\theta \int_{-\infty}^{\infty} dp\rho\left(\tau,\theta,p\right)=1, \quad \forall\tau. \tag{3}$$

As shown in (Liapzev 2019) (see also (Kuznetsov 2006)), the probability density satisfies the partial differential equation of the first order. In particular, for the system considered in this paper, the equation has the form:

$$\frac{\partial\rho}{\partial\tau}+p\frac{\partial\rho}{\partial\theta}+f\sin\theta\cos\tau\frac{\partial\rho}{\partial p}-\gamma\frac{\partial\rho}{\partial p}=0. \tag{4}$$

Note that the density of the probability distribution can be approximately obtained by a numerical experiment. To do this, note that the variables τ and θ are limited to the range [0, 2π]. It is not difficult to show (Liapzev 2019) that for any parameter values, the variable p is limited to a certain range [$-p_{\max}$, $p_{\max}$]. To calculate the probability density, each of the variable intervals can be divided into identical cells. Denote the corresponding numbers of cells by N_τ, N_θ and N_p. As a result, the entire region of the phase space is divided into $N_\theta N_p N_\tau$ identical cells. Let τ_i be the values of the variable τ, lying in the middle of each of the N_τ intervals. To numerically find the probability density, some initial values θ (0) and p (0) are selected and then numerical solutions θ (t_j), p (t_j) are found, where the values of t_j lie in the interval [0, dT] and are equal in modulus 2π to the values of τ_i. For each of the t_j values, a cell in the phase space is determined, into which the values θ (t_j), p (t_j) and τi fall, as a result of which the values of the numbers of points in each of the cells are calculated. Next, the values θ (dT) and p (dT) are chosen as the initial ones, and solutions θ (t_j), p (t_j) are found, where the values of t_j lie in the interval [dT, $2dT$], as a result of which the numbers of points in each of the cells are supplemented. After repeating such iterations many times and after normalization to one, the probability density is found in the form of an array ρ (τ_i, θ_j, p_k).

Calculation of the dimension and entropy of a chaotic attractor

To calculate entropy, we use the formula (for details, see, for example, (Malinetsky, Potapov 2000)):

$$S=-\sum_i w_i \ln\left(w_i\right), \tag{5}$$

where the index i is the cell number, w_i is the probability that the system will be in the region of the phase space bounded by the i–th cell. The entropy determined this way depends on the size of the cells, and, when these sizes tend to zero, (the number of cells tends to infinity) it has no finite limit. Indeed, the probability of finding a system in cell i is expressed in terms of probability density:

$$w_i = \rho_i\,\Delta V,$$

where ρ_i is the probability density for the i^{th} cell and ΔV is the volume of the cell. If the characteristic cell size (in one dimension) is ε, then with the dimension of the phase space n, the cell volume is $\Delta V \approx \varepsilon^n$ and for entropy we obtain the expression:

$$S \cong -\sum_i \rho_i\,\Delta V\,(\ln\rho_i + \ln\Delta V) = -\sum_i w_i \ln\rho_i - \sum_i p_i \ln(\varepsilon^n) = -\langle \ln\rho\rangle - n\ln\varepsilon.$$

The first term has a finite limit at $\varepsilon \rightarrow 0$, let us denote it by S_0; the second term increases as a logarithm. Thus, for small values of ε, the expression for entropy can be written as:

$$S \approx S_0 - n\ln\varepsilon. \tag{6}$$

Expression (6) also turns out to be valid in the case when the dimension of the set n, in which the phase trajectory lies, turns out to be fractional, which occurs when the state of the system tends to a chaotic attractor.

Note that in numerical calculations, the tendency of $\varepsilon \to 0$ is provided by the tendency of a number of cells to infinity. In particular, in the case we are considering, we can put $N_\theta = N_p = N_\tau = N$. Then the dependence ε on N can be represented as: $\varepsilon = \varepsilon_0/N$, where ε_0 is a constant. As a result, the expression for entropy can be written as:

$$S(N) \approx S_0 - n\ln\varepsilon_0 + n\ln N. \tag{7}$$

Expression (7) can be used to numerically determine the dimension of a chaotic attractor. For a given value of N, the probability p_i can be defined as the number of points in i cell relative to the number of points in all cells. Using formula (5) we obtain the entropy $S(N)$. By varying the value of N, one can get the dependence $S(\ln N)$. For large values of N, the graph should approach a straight line, the tangent of the inclination angle of which is equal to the dimension of the attractor n.

Note that definition (5) for the entropy of the attractor can be extended to the case of regular motion, when the attractor is a limit cycle. The trajectory in phase space is a line, a point on which can be characterized by the value τ (Fig. 1). The period of regular movement is a multiple of the period of external influence. Now let the entire phase space be divided into N^3 cells (N in each dimension). With a multiplicity equal to q, each value of τ_i corresponds to q cells through which the phase trajectory passes. Thus, for one period, Nq cells are filled with one point, and for M periods, each of these cells will be filled with M points. The probability that the system is located at one of these points is equal to $w = \dfrac{M}{MNq} = \dfrac{1}{Nq}$. The probability of being at any other point is zero. Thus,

$$S(N) = -\sum_{i=1}^{Nq} \frac{1}{Nq} \ln\left(\frac{1}{Nq}\right) = \ln q + \ln N \ .$$

This expression coincides with expression (7), if we take into account that the dimension of the attractor is n=1 and put:

$$S_0 - n\ln\varepsilon_0 = \ln q. \tag{8}$$

In this paper, the dimension of the chaotic attractor was calculated as follows. In each dimension (variables τ, θ and p), the range of acceptable values was divided into N = 90 identical intervals. Thus, the phase space was divided into 729.000 cells. The time intervals at which the system of equations (2) was solved were chosen equal to $dT = 20\pi$, that is, 10 periods of external influence. At each interval, a three-dimensional array ρ (N, τ_i, θ_j, p_k) with a dimension of 90×90×90 was filled. The calculation was carried out over 10,000 dT intervals. At the same time, by combining 8 nearest cells (2 in each dimension), an array ρ ($N/2$, τ_i, θ_j, p_k) with dimension 45×45×45 was calculated and by combining 9 nearest cells of the array ρ (N, τ_i, θ_j, p_k) (3 in each dimension), an array was calculated ρ ($N/3$, τ_i, θ_j, p_k) with dimension 30×30×30. Further, using these arrays, the entropies $S(N)$, $S(N/2)$, $S(N/3)$ were calculated using formula (5). The calculation shows that the linear dependence of entropy on *ln* (formula (7)) is performed in a good approximation. On the $S(\ln N)$ graph, the points lie almost on the same straight line, and the values of the slope angle tangents calculated from the points with N = 90 and N = 45 differ from the values calculated from the points with N = 90 and N = 30 by about 1%. In this paper, the calculation was carried out for a parameter characterizing the dissipation of γ = 0.1 in the range of values of the control parameter $f \in$ [2.159, 22.61], for a value of N = 90. In the case of chaotic motion, the average values of $S(90)$ and n were approximately equal to 12 and 2.5, respectively. However, in some areas of the values of f, significant deviations from these average values were observed, which made it possible to calculate the values of the parameters S_0 and ε_0 that are included into expression (7).

Dependence of the entropy and dimension of the attractor on the control parameter

As mentioned above, the areas of values of the control parameter, at which chaotic motion is realized, alternate with areas of regular motion. In the case of a regular solution, the numerical calculation of entropy is fully consistent with expression (8). The results of the calculation in one of the regions of chaotic motion $f \in$ [2.159,11.28] are shown in Fig. 3.

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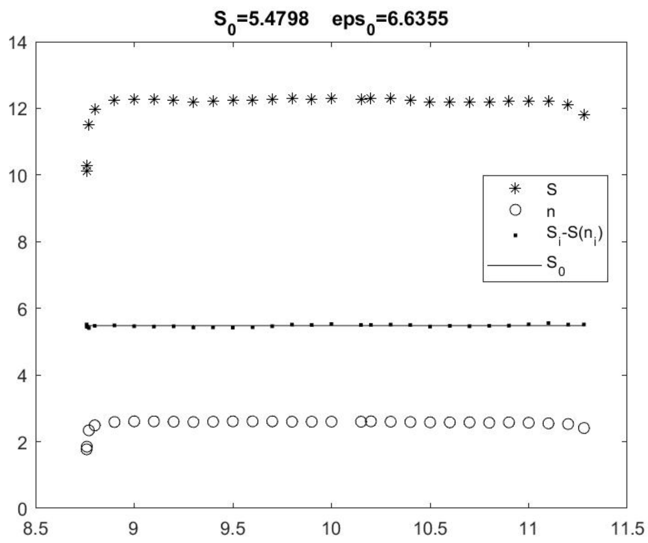

Fig. 3. Calculated values of entropy (upper graph) and dimension of the attractor (lower graph) depending on the control parameter

As can be seen from the figure, significant (up to 30%) changes in the values of S and n are observed at the border of the plot (upper and lower graphs). These changes are due to the fact that at the borders there is a transition from chaotic movement to regular movement through intermittency. This means that during sufficiently large time intervals (hundreds of periods of external influence), the movement is periodic, after which there is a transition to chaotic movement. Similar transitions between order and chaos are observed in various nonlinear systems (Grinchenko et al. 2017). A simultaneous decrease in the values of n and S at the boundary of the interval suggests that a decrease in entropy is due to a decrease in the dimension of the attractor in accordance with expression (7). This means that the values of $\{F_i, S_i, n_i\}$ obtained as a result of the numerical experiment can be approximated by the expression:

$$S_i = S_0 - n_i \ln(\varepsilon_0) + n_i \ln N, \tag{9}$$

where the i index numbers the points on the graph (Fig. 3), and according to the calculations $N = 90$. Using the least squares method, it is possible to find the parameters S_0 and ε_0, at which the results of the numerical experiment are best described by expression (9). The least squares method also allows one to estimate the error with which the parameters are calculated. The results of such calculations for several ranges of values of f, within which the movement is chaotic, are shown in Table 1.

Table 1. Estimation of the parameters ε_0 and S_0 (formula 9) based on the results of a numerical experiment

Range		S_0	δS_0 (%)	ε_0	$\delta\varepsilon_0$ (%)
f_{min}	f_{max}				
2.165	3.76	3.4	15	3.1	38
7.471	8.32	5.8	5	6.9	6
8.77	11.28	5.5	5	6.8	6
16.875	18.794	5.5	9	6.7	11
19.35	22.61	5.9	6	7.8	7

The results of such an approximation for the range $f \in [2.159, 11.28]$ are clearly shown in Fig. 3. Expression (9) can be represented as: $S_0 = S_i + n_i \ln(\varepsilon_0) - n_i \ln N$. The points on the middle graph correspond to the values $S_i + n_i \ln(\varepsilon_0) - n_i \ln N$ for the calculated value ε_0, and the straight line near which the points are located corresponds to the calculated value S_0.

Discussion of the results

As follows from Table 1, the values S_0 and ε_0 increase with an increase in the control parameter f. Despite the fact that, in our opinion no rigorous theory that allows calculating the values of S_0 and ε_0 can be constructed, there can be some theoretical estimates that explain the tendency of these values to increase and give values close to the values of the numerical experiment.

The value S_0, in the sense of its definition, should not depend on the properties of the fractal inherent in the chaotic attractor, but should be determined by some integral characteristics of the probability density for the chaotic attractor. To determine these properties, one can try to refer to the systems studied by statistical physics (see, for example, (Landau, Lifshitz 1980)). For such systems, entropy can be defined by the expression:

$$S = \ln\Delta\Gamma. \tag{10}$$

In this expression, $\Delta\Gamma$ is a statistical weight, the meaning of which is that it characterizes the size of the region of the phase space in which this system spends almost all the time. Since the definition of entropy should not depend on the units of measurement, the dimensions of the specified area of space should be calculated relative to some dimensional quantity taken as a unit of measurement. The quasi-classical limit gives as such a value of the Planck constant h, and for $\Delta\Gamma$ the value:

$$\Delta\Gamma = \frac{\prod_{i=1}^{s} \Delta q_i \Delta p_i}{h^s}, \tag{11}$$

where s is the number of degrees of freedom of the system, q_i and p_i are the generalized coordinates and impulses of the system.

Without pretending to be a strict justification, we generalize the above formulas for our case. We define the part of entropy that does not depend on the dimension of the chaotic attractor by the expression:

$$S_0 = \ln\Delta\Gamma. \tag{12}$$

Taking into account the large-scale time transformations carried out, the variables τ, θ and p are already measured in relative units. Taking expression (11) as a basis, we define the statistical weight by the expression:

$$\Delta\Gamma = \Delta\tau\Delta\theta\Delta p, \tag{13}$$

where $\Delta\tau$, $\Delta\theta$, Δp are the characteristic sizes of the regions of the corresponding variables in which the system is located most of the time. Obviously, $\Delta\tau = \Delta\theta = 1$ for any value of the control parameter f. As for the value Δp, then, as the calculation shows, it increases with an increase in the parameter f, since, in general, the movement becomes more intense. It is hardly possible to propose an analytical estimate of the dependence $\Delta p(f)$, however, using numerical results for the probability density of this system, it is possible to obtain numerical estimates which will be given below.

To estimate the value of ε_0, we note that in the sense of this value, it must be associated with a region of the phase space in which the values of τ, θ and p change. But for any value of the control parameter f corresponding to chaotic motion, the values τ and θ always lie in the range $[0, 2\pi]$. Only the region of the variable p, as the numerical experiment shows, increases with the increasing f.

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To get an approximate estimate of ε_0, the unit orts $\{e_\theta, e_p, e_\tau\}$ should be reduced to one scale so that $e'_\theta = e'_p = e'_\tau = \varepsilon_0$. When choosing a scale, it is advisable, in accordance with formula (8), in the case of the simplest regular motion, to put $S_0 = 0$ and, consequently, $\varepsilon_0 = 1$. Since in this case the variable τ takes a value in the range $[0, 2\pi]$, the scale transformation has the form:

$$e'_\tau = \pi e_\tau \tag{14}$$

It is reasonable to assume that such a transformation persists in chaotic motion, and is also true for the variable θ, which takes values in the interval $[0, 2\pi]$ in chaotic motion:

$$e'_\theta = \pi e_\theta. \tag{15}$$

But, since the variables θ, p and τ are connected by a system of equations (2), the scale transformations (14) and (15) result in a scale transformation for the variable p:

$$e'_p = e_p.$$

From these arguments it follows that to estimate the value of ε_0, one can take half of the range of changes in the value of p in chaotic motion. As mentioned above, for any value of the control parameter f, the area of change of p is limited by some interval $[-p_{\max}, p_{\max}]$. Thus, to estimate the value of ε_0, we can put:

$$\varepsilon_0 = p_{\max}. \tag{16}$$

Numerical estimates of the values Δp and $p_{\max}$ can be obtained using the calculated array $\rho(N, \tau_i, \theta_j, p_k)$. To do this, we average the probability density $\rho(N, \tau_i, \theta_j, p_k)$ over the variables τ and θ:

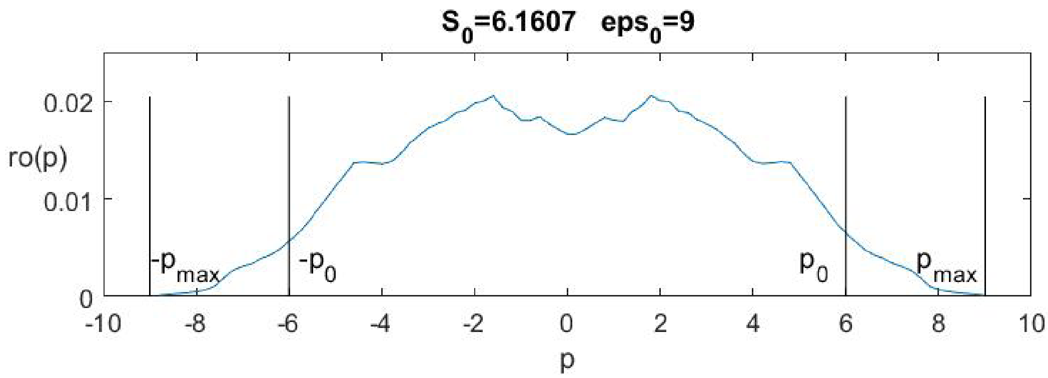

Fig. 4. Graph of the averaged probability density $\rho(N, p_k)$

From the definition of probability density, the area under the graph is equal to one. Let us define $\Delta p = 2p_0$, where p_0 is chosen so that the area under the graph between the values $-p_0$ and p_0 is approximately 0.9. Approximately, we can assume that the system spends almost all the time in the region $[-p_0, p_0]$.

The comparison of the results obtained by the numerical experiment (Table 1) and estimates according to formulas (13) and (16) are clearly shown in Fig. 5. Estimates are obtained for the values of the control parameter f lying in the middle of the corresponding range.

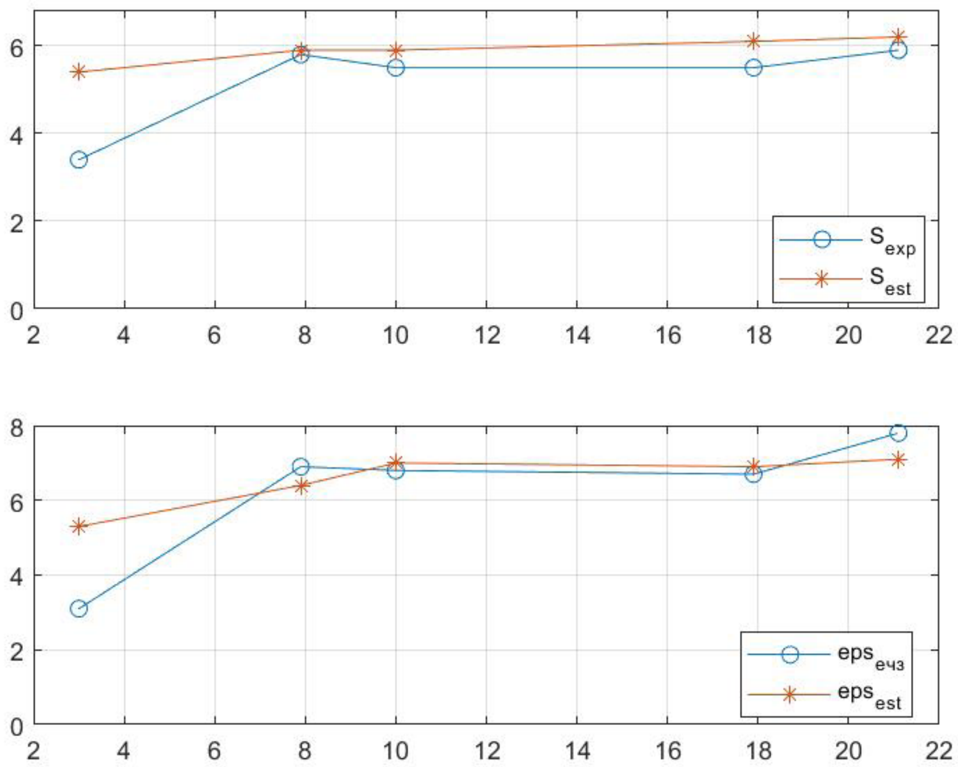

Fig. 5. Comparison of the results of the numerical experiment and the estimates obtained by formulas (13) and (16) for the parameters S_0 and ε_0

As can be seen from the figure, with sufficiently large values of the control parameter, rough estimates turn out to be close to the values obtained by the numerical experiment.

Conflict of interest

The author declares that there is no conflict of interest, either existing or potential.

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