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Reduced integral representations for the probabilities of photon emission in a constant external electric field

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Abstract. In the case of an intense external electric field, there exist many transition channels corresponding to the violation of the vacuum stability. It was shown that depending on the setting of a problem there is a number of integral representations for the probabilities of one photon emission due to a constant electric field. These representations are the Fourier transformations of the product of two Weber parabolic cylinder functions that are solutions of the same differential equation. To simplify the study of the probabilities, we expressed such a Fourier transformation via the confluent hypergeometric function.

Keywords: Photon emission, quantum electrodynamics, strong external field, Schwinger effect, Fourier transformation

Introduction

In quantum electrodynamics with strong electric-like external fields (strong-field QED) there exists a vacuum instability due to the effect of real particle creation from the vacuum caused by external fields—the Schwinger effect (Schwinger 1951). A number of publications, reviews, and books are devoted to the effect itself and to developing different calculation methods in theories with unstable vacuum, see Refs. (Birrell, Davies 1982; Fedotov et al. 2022; Fradkin et al. 1991; Gelis, Tanji 2016; Greiner 1985; Grib et al. 1994; Ruffini et al. 2010) for a review. Until recently, problems related to particle creation from the vacuum were of a purely theoretical interest. This is related to the fact that, due to the presence of large gaps between the upper and the lower branches in the spectrum of the electron-positron pair, particle creation effects can be observed only in huge external electric fields of the magnitude of $E_c = m^2/e \simeq 10^{16}\, V/cm$. However, recent technological advances in laser science suggest that lasers may be able to reach the nonperturbative regime of pair production in the near future; see, e. g., the review (Fedotov et al. 2022) and references therein. Moreover, the situation has changed completely in recent years regarding applications to condensed matter physics: simulation of particle creation by external fields has become an observable effect in physics of graphene and similar nanostructures (say, in topological insulators and Weyl semimetals); see, e. g., the reviews (Sarma et al. 2011; Vafek, Vishwanath 2014). This is explained by the fact that low-energy single-electron dynamics in graphene monolayers at the charge neutrality point is described by the Dirac model, being a 2 + 1 dimensional version of massless QED with the Fermi velocity $v_F \simeq 10^6$ m/s playing the role of the speed of light in relativistic particle dynamics (the reduced QED $_{3,2}$).

Due to the recent detection of an optical radiation in the graphene accompanying the creation of electron-hole pairs by a terahertz electric pulse (Oladyshkin et al. 2017), it becomes possible to make a comparison of the corresponding theoretical calculations with experiments. This terahertz pulse can be considered as slowly varying and locally approximated by a constant electric field. We see that the theory of photon emission under the action of a strong constant external electric field is of interest. Processes involving photon emission and annihilation in the presence of the vacuum instability are processes of higher order in radiative corrections. Adequate nonperturbative calculations with respect to the external field can be done using a general approach to QED with strong external fields (Fradkin et al. 1991) (based on the existence of special exact solutions of the Dirac equation with this field). The study of these processes in details is technically complicated. It can be seen from the number of calculations for particular cases: the emission of a photon in the case of a small instability in 3 + 1 QED (Nikishov 1971) and in graphene (Yokomizo 2014) and the probability of one photon emission from a single-electron state in the presence of a strong field in graphene (Aslyamova, Gavrilov 2020; Gavrilov, Gitman 2017). In the case of an intense external field, there exist many transition channels corresponding to the violation of the vacuum stability. By this reason, depending on the setting of a problem there is a number of integral representations for the probabilities of one photon emission due to a constant electric field. All of these representations are Fourier transformations of the product of two Weber parabolic cylinder functions (WPCFs) of some kind. Some of these integral representations with particular WPCFs were studied by A. I. Nikishov (Nikishov 1971). In the case of a strong field, the total probability of radiative processes are of interest. Therefore, integral representations with the number of other pairs of WPCFs for the probabilities are needed; see sections 3 and 4 in the book (Fradkin et al. 1991). In this article we study general properties of these representations that are quite similar in 3 + 1 QED and the reduced QED$_{3,2}$. Bearing in mind application of the obtained results in the framework of the theory of photon emission in nanostructures, we use representations of the QED$_{3,2}$ in what follows.

Probabilities of emission

Adjustment of the general probability representations (Fradkin et al. 1991) to the reduced QED$_{3,2}$ to describe one species of the Dirac fermions (small mass m and particle charge $q = -e$, $e > 0$ is the absolute value of the electron charge) in the graphene interacting with a constant external electric field $E > 0$ and photons is presented in Ref. (Gavrilov, Gitman 2017). We use notation $\overleftarrow{\mathbf{k}} = \left(\mathbf{k}, k_z\right)$ for three-dimensional wave vector of a photon. The two-dimensional vector k = (k_x, k_y) is a projection of $\overleftarrow{\mathbf{k}}$ on the graphene plane, $\omega = c\left|\overleftarrow{\mathbf{k}}\right|$, $\vartheta = 1{,}2$ denotes a polarisation index, and ε is the relative permittivity (for the graphene suspended in a vacuum $\varepsilon = 1$).

The probability of one photon emission with given $\overleftarrow{\mathbf{k}}$ and ϑ from a single-electron (hole) state with given two-dimensional momentum p = (p_x, p_y) per unit frequency and solid angle is expressed as follows:

$$\frac{dP\left(\overleftarrow{\mathbf{k}}, \vartheta \middle| \overset{\pm}{\mathbf{p}}\right)}{d\omega d\Omega} = \frac{\alpha}{\varepsilon}\left(\frac{v_F}{c}\right)^2 \frac{\omega \Delta t_{st}^2}{\left(2\pi\right)^2} \left|M_{\mathbf{p}'\mathbf{p}}^{\pm}\right|^2 \Big|_{\mathbf{p}'=\mathbf{p}-\hbar\mathbf{k}} ,$$

$$M_{\mathbf{p}'\mathbf{p}}^{\pm} = \mp \frac{S}{\Delta t_{st}} \int_{-\infty}^{\infty} {}_{\pm}\bar{\Psi}_{\mathbf{p}'}\left(t\right) \gamma \epsilon_{\overleftarrow{\mathbf{k}}\vartheta} \; {}_{\pm}\Psi_{\mathbf{p}}\left(t\right) e^{i\omega t} dt , \quad (1)$$

where $\Delta t_{st} = (|eE|\, v_F/\hbar)^{-1/2} \gg t_\gamma$ is a big characteristic time scale and t_γ is the microscopic time scale, $\epsilon_{\overleftarrow{\mathbf{k}}\vartheta}$ are mutual orthogonal unit polarisation vectors transversal to three-dimensional wave vector $\overleftarrow{\mathbf{k}}$, and $\gamma = (\gamma^1, \gamma^2)$ are 2 × 2 the Dirac γ -matrices,

$\gamma^0 = \sigma^3$, $\gamma^1 = i\sigma^2$, $\gamma^2 = -i\sigma^1$,

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where the σ^j are Pauli matrices. Solutions of the Dirac equation in 2 + 1 dimensions with a constant field for the case of graphene physics were studied in details in Ref. (Gavrilov, Gitman 1996). It was demonstrated that the corresponding in-set $\{_\zeta\psi_n(t,\boldsymbol{r})\}$ and out-set $\{^\zeta\psi_n(t,\boldsymbol{r})\}$ can be chosen in the form:

$$
\begin{aligned}
{}_\pm\Psi_{\mathbf{p}}(t,\mathbf{r}) &= \left(i\hbar\partial_t + H^{\text{ext}}\right) {}_\pm\Phi_{\mathbf{p},\pm1}(t,\mathbf{r}), \quad {}_\pm\Phi_{\mathbf{p},\pm1}(t,\mathbf{r}) = e^{i\mathbf{p}\cdot\mathbf{r}/\hbar}\ {}_\pm\varphi_{\mathbf{p},\pm1}(t)U_{\pm1}, \\
{}^\pm\Psi_{\mathbf{p}}(t,\mathbf{r}) &= \left(i\hbar\partial_t + H^{\text{ext}}\right) {}^\pm\Phi_{\mathbf{p},\mp1}(t,\mathbf{r}), \quad {}^\pm\Phi_{\mathbf{p},\mp}(t,\mathbf{r}) = e^{i\mathbf{p}\cdot\mathbf{r}/\hbar}\ {}^\pm\varphi_{\mathbf{p},\mp1}(t)U_{\mp1}, \\
H^{\text{ext}} &= v_F\gamma^0\left(\gamma^1\left(p_x - eEt\right) + \gamma^2 p_y + mv_F\right),
\end{aligned}
\tag{2}
$$

where U_s are constant orthonormalised spinors

$$
U_{+1} = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}, \quad U_{-1} = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\end{pmatrix}.
$$

The functions ${}_\pm\phi_{p,s}(t)$ and ${}^\pm\phi_{p,s}(t)$ have the form of the WPCFs:

$$
\begin{aligned}
{}^{-}_{+}\varphi_{\mathbf{p},s}(t) &= CD_{\nu-\frac{1+s}{2}}\left[\pm(1-i)\xi\right], \quad {}^{+}_{-}\varphi_{\mathbf{p},s}(t) = CD_{-\nu-\frac{1-s}{2}}\left[\pm(1+i)\xi\right], \quad \nu = \frac{i\lambda}{2}, \\
\xi &= \sqrt{\frac{v_F}{eE}}\left(eEt - p_x\right), \quad \lambda = \frac{v_F p_y^2 + m^2 v_F^3}{eE}, \quad C = \left(2eE\hbar v_F S\right)^{-1/2}\exp\left(-\frac{\pi\lambda}{8}\right),
\end{aligned}
\tag{3}
$$

where S is the graphene area. An in-state ${}_\pm\psi_p(t,\boldsymbol{r})$ describes a particle/hole with a well-defined energy at the distant past. Similarly, an out-state ${}^\pm\psi_p(t,\boldsymbol{r})$ describes a particle/hole with a well-defined energy at the distant future.

The probability of the one-photon emission with given $\overleftarrow{\mathbf{k}}$ and ϑ, which accompanies the production from the initial vacuum state of pairs of charged species with a given momentum $\mathbf{p}$ per unit frequency and solid angle reads:

$$
\begin{aligned}
\frac{dP\left(\mathbf{p};\mathbf{K},\vartheta|0\right)}{d\omega d\Omega} &= \frac{\alpha}{\varepsilon}\left(\frac{v_F}{c}\right)^2 \frac{\omega\Delta t_{st}^2}{(2\pi)^2}\left|M^0_{\mathbf{p}'\mathbf{p}}\right|^2\Big|_{\mathbf{p}'=\mathbf{p}-\hbar\mathbf{k}}, \\
M^0_{\mathbf{p}'\mathbf{p}} &= -\frac{S}{\Delta t_{st}}\int_{t_1}^{t_2} {}_+\bar{\Psi}_{\mathbf{p}'}(t)\boldsymbol{\gamma}\epsilon_{\mathbf{K}\vartheta}\ {}_-\Psi_{\mathbf{p}}(t)e^{i\omega t}dt.
\end{aligned}
\tag{4}
$$

By inserting equation (3) into equation (2) and explicitly taking derivatives, we find that both $M^\pm_{\mathbf{p}'\mathbf{p}}$ and $M^0_{\mathbf{p}'\mathbf{p}}$ consist of the superposition of the following integrals:

$$
\begin{aligned}
Y_{j'j} &= \int_{-\infty}^{\infty} D_{-\nu'-j'}\left[-(1+i)(u-u_x/2)\right]D_{\nu-j}\left[-(1-i)(u+u_x/2)\right]e^{iu_0u}du, \\
\tilde{Y}_{j'j} &= \int_{-\infty}^{\infty} D_{-\nu'-j'}\left[-(1+i)(u-u_x/2)\right]D_{-\nu-j}\left[-(1+i)(u+u_x/2)\right]e^{iu_0u}du,
\end{aligned}
\tag{5}
$$

where

$$u = \sqrt{\frac{v_F}{eE\hbar}}\left[eEt - \frac{1}{2}\left(p_x + p_x^{'}\right)\right],$$

$$u_x = \sqrt{\frac{v_F}{eE\hbar}}\left(p_x^{'} - p_x\right), \quad u_0 = \Delta t_{st}\omega \quad . \tag{6}$$

Note that the probabilities of one-photon emission and absorption in any initial or final state can be expressed in similar representations with a pair of appropriate spinors $_{\pm}\psi_p\,(t, \boldsymbol{r})$ and $^{\pm}\psi_p\,(t, \boldsymbol{r})$.

Reduced integral representation

Integrals (5) can be simplified using the hyperbolic coordinates ρ and ϕ,

$$\rho = \sqrt{\left|u_0^2 - u_x^2\right|}, \quad \tanh\varphi = \frac{u_x}{u_0} \text{ if } u_0^2 - u_x^2 > 0 \ ,$$

$$\tanh\varphi = \frac{u_0}{u_x} \text{ if } u_0^2 - u_x^2 < 0 \ . \tag{7}$$

By applying Nikishov's approach (Nikishov 1971) we consider a more general case. Note that integrals (5) represent particular cases of the more general integrals

$$J_{\Lambda'\Lambda}^{\zeta'\zeta}\left(\rho,\varphi\right) = \int_{-\infty}^{+\infty} du\, f_{\Lambda'}^{\zeta'}\left(u - u_x/2\right) f_{\Lambda}^{\zeta}\left(u + u_x/2\right) e^{iu_0 u}\ , \tag{8}$$

where $f_{\Lambda}^{\zeta}\left(z\right)$ are WPCFs satisfying the differential equation

$$\left(\frac{d^2}{dz^2} + z^2 + \Lambda\right) f_{\Lambda}^{\zeta}\left(z\right) = 0\ , \tag{9}$$

and u_0 and u_x, defined by equation (6), are:

$$u_0 = \rho\cosh\varphi\,,\ u_x = \rho\sinh\varphi \ \text{ if } u_0^2 > u_x^2,$$

$$u_0 = \rho\sinh\varphi\,,\ u_x = \rho\cosh\varphi \ \text{ if } u_0^2 < u_x^2 \text{ and } u_x > 0, \tag{10}$$

$$u_0 = -\rho\sinh\varphi\,,\ u_x = -\rho\cosh\varphi \ \text{ if } u_0^2 < u_x^2 \text{ and } u_x < 0.$$

The functions $f_{\Lambda}^{\zeta}\left(z\right)$ with different values of $\zeta = \pm$ are two linearly independent solutions of equation (9) with some complex parameters Λ. In particular,

$$J_{\Lambda'\Lambda}^{-+}\left(\rho,\varphi\right) = Y_{j'j},\ \ \Lambda = \lambda + i\left(2j-1\right),\ \ \Lambda' = \lambda' + i\left(1-2j'\right),$$

$$J_{\Lambda'\Lambda}^{--}\left(\rho,\varphi\right) = \tilde{Y}_{j'j},\ \ \Lambda = \lambda + i\left(1-2j\right),\ \ \Lambda' = \lambda' + i\left(1-2j'\right). \tag{11}$$

Calculating the derivative of integral (8) with respect to the hyperbolic angle ϕ, we find:

$$\frac{\partial J^{\zeta'\zeta}_{\Lambda'\Lambda}(\rho,\varphi)}{\partial\varphi}=W+\int_{-\infty}^{+\infty}iu_x f^{\zeta'}_{\Lambda'}(u-u_x/2)\,f^{\zeta}_{\Lambda}(u+u_x/2)\,e^{iu_0u}du\ ,$$

$$W=\frac{u_0}{2}\int_{-\infty}^{+\infty}\left[f^{\zeta'}_{\Lambda'}(u-u_x/2)\left.\frac{\partial f^{\zeta}_{\Lambda}(z)}{\partial z}\right|_{z=u+u_x/2}\right.$$

$$\left.-\left.\frac{\partial f^{\zeta'}_{\Lambda'}(z)}{\partial z}\right|_{z=u-u_x/2}f^{\zeta}_{\Lambda}(u+u_x/2)\right]e^{iu_0u}du\ ,\ u_x=\frac{\partial u_0}{\partial\varphi}\ ,\ u_0=\frac{\partial u_x}{\partial\varphi}\ .$$

Integrating by parts and neglecting boundary terms, we can transform W into the following form:

$$W=\frac{i}{2}\int_{-\infty}^{+\infty}\left[f^{\zeta'}_{\Lambda'}(u-u_x/2)\left.\frac{\partial^2 f^{\zeta}_{\Lambda}(z)}{\partial z^2}\right|_{z=u+u_x/2}\right.$$
$$\left.-\left.\frac{\partial^2 f^{\zeta'}_{\Lambda'}(z)}{\partial z^2}\right|_{z=u-u_x/2}f^{\zeta}_{\Lambda}(u+u_x/2)\right]e^{iu_0u}du\ . \tag{12}$$

Using equation (9) in integral (12), we find:

$$\frac{\partial J^{\zeta'\zeta}_{\Lambda'\Lambda}(\rho,\varphi)}{\partial\varphi}=\frac{i}{2}(\Lambda'-\Lambda)\,J^{\zeta'\zeta}_{\Lambda'\Lambda}(\rho,\varphi). \tag{13}$$

Solutions of this equation are:

$$J^{\zeta'\zeta}_{\Lambda'\Lambda}(\rho,\varphi)=e^{\frac{i}{2}(\Lambda'-\Lambda)(\varphi-\varphi_0)}J^{\zeta'\zeta}_{\Lambda'\Lambda}(\rho,\varphi_0), \tag{14}$$

where $\phi_0 = 0$ if $u_0^2 > u_x^2$ and we choose an appropriate value for ϕ_0 (sign of ϕ_0 is the same with u_x) if $u_0^2 < u_x^2$. The ϕ dependence of integrals (8) can be factorised with the help of equation (14). In a more interesting case, $u_0^2 > u_x^2$, we get:

$$Y_{j'j}=\exp\left[\left(i\frac{\lambda'-\lambda}{2}+j'+j-1\right)\varphi\right]J_{j',j}(\rho),$$
$$J_{j',j}(\rho)=\int_{-\infty}^{\infty}D_{-\nu'-j'}\left[-(1+i)u\right]D_{\nu-j}\left[-(1-i)u\right]e^{i\rho u}du; \tag{15}$$

$$\tilde{Y}_{j'j}=\exp\left[\left(i\frac{\lambda'-\lambda}{2}+j'-j\right)\varphi\right]\tilde{J}_{j',j}(\rho),$$
$$\tilde{J}_{j',j}(\rho)=\int_{-\infty}^{\infty}D_{-\nu'-j'}\left[-(1+i)u\right]D_{-\nu-j}\left[-(1+i)u\right]e^{i\rho u}du. \tag{16}$$

If $u_0^2 > u_x^2$, we assume that ϕ_0=0 and use the notation $J^{\zeta'\zeta}_{\Lambda'\Lambda}(\rho)=J^{\zeta'\zeta}_{\Lambda'\Lambda}(\rho,0)$ in what follows. This function satisfies the differential equation (Nikishov 1971)

$$\left[\frac{d^2}{d\rho^2}+\frac{1}{\rho}\frac{d}{d\rho}+\frac{(\Lambda-\Lambda')^2}{4\rho^2}+\frac{\rho^2}{4}-\frac{\Lambda+\Lambda'}{2}\right]J^{\zeta'\zeta}_{\Lambda'\Lambda}(\rho)=0\ . \tag{17}$$

This fact can be verified by performing integrations by parts taking into account equation (9). The differential equation (17) can be reduced to a confluent hypergeometric equation. Using two linearly independent solutions of such an equation, we find general solution of the differential equation (17)

$$J_{\Lambda'\Lambda}^{\zeta'\zeta}\left(\rho\right)=e^{-\eta/2}\left[C_1\eta^{i\beta}\Phi\left(\frac{i\Lambda}{2}+\frac{1}{2},1+2i\beta;\eta\right)+C_2\eta^{-i\beta}\Phi\left(\frac{i\Lambda'}{2}+\frac{1}{2},1-2i\beta;\eta\right)\right],$$
$$\eta=-i\rho^2/2,\quad \beta=\left(\Lambda-\Lambda'\right)/4\ , \tag{18}$$

where the C_1 and C_2 are some undetermined coefficients, which must be fixed by appropriate boundary conditions so that solution (18) corresponds to the original integral (8).

The confluent hypergeometric function $\Phi(a,c;\eta)$ is entire in η and a, and is a meromorphic function of c. Note that $\Phi(a,c;0) = 1$. WPCFs are entire functions of Λ and Λ'. We can see that the integrals $J_{\Lambda'\Lambda}^{\zeta'\zeta}\left(\rho\right)$ are entire functions of Λ and Λ' and meromorphic functions of Λ – Λ'. Then we can find a boundary condition $J_{\Lambda'\Lambda}^{\zeta'\zeta}\left(\rho\right)$ at $\rho \to 0$ for some convenient values of *j* and *j'*. The remaining integrals $J_{\Lambda'\Lambda}^{\zeta'\zeta}\left(\rho\right)$ can be obtained by extending domains of Λ and Λ' by an analytic continuation.

Let us start with $\tilde{J}_{0,0}\left(\rho\right)$ given by equation (16). This integral can be represented as a solution of equation (17) where $\Lambda' = \lambda' + i$ and $\Lambda = \lambda + i$. The coefficients C_1 and C_2 in equation (18) can be fixed by a comparison with the $\rho\to 0$ limit of integral (16). Let us first represent this integral as follows:

$$\tilde{J}_{0,0}\left(\rho\right)=F^0+F^++F^-,\ F^+=\int_0^\infty f^+\left(u\right)e^{i\rho u}du,\ F^-=\int_{-\infty}^0 f^-\left(u\right)e^{i\rho u}du\ ,$$
$$F^0=\int_0^\infty f\left(u\right)\left[f\left(u\right)-f^+\left(u\right)\right]e^{i\rho u}du+\int_{-\infty}^0 f\left(u\right)\left[f\left(u\right)-f^-\left(u\right)\right]e^{i\rho u}du\ , \tag{19}$$
$$f\left(u\right)=D_{-\nu'}\left[-\left(1+i\right)u\right]D_{-\nu}\left[-\left(1+i\right)u\right]\ ,$$

where $f^\pm(u) = f(u)|_{u\to\pm\infty}$. It can be seen that function (18) is reduced to the oscillations $C_1\eta^{i\beta} + C_2\eta^{-i\beta}$ as $\rho \to 0$. Then ρ -independent terms do not contribute to the integrals F^0 and $F^\pm$. Taking into account that $\lim_{(\rho\to 0)}F^0$ and $\lim_{(\rho\to 0)}F^-$ do not depend on ρ, we see that the oscillation terms of F^+ are only essential. Using Bateman's relatio ns (8.2.(7)) and (8.4.(1)) (Bateman 1953), we find:

$$\tilde{J}_{0,0}\left(\rho\right)=\sqrt{\pi}e^{i\pi\left(\nu+\nu'-1\right)/4}\left[e^{i\pi\nu/2}\frac{\Gamma\left(\nu-\nu'\right)}{\Gamma\left(\nu\right)}\left(\frac{\rho}{\sqrt{2}}\right)^{\nu'-\nu}+e^{i\pi\nu'/2}\frac{\Gamma\left(\nu'-\nu\right)}{\Gamma\left(\nu'\right)}\left(\frac{\rho}{\sqrt{2}}\right)^{\nu-\nu'}\ \text{as}\ \ \rho\to 0\right]. \tag{20}$$

Comparing equations (18) and (20), we obtain:

$$C_1=\sqrt{\pi}e^{i\pi\left(\nu+\nu'-1/2\right)/2}\frac{\Gamma\left(\nu'-\nu\right)}{\Gamma\left(\nu'\right)},\ C_2=\sqrt{\pi}e^{i\pi\left(\nu+\nu'-1/2\right)/2}\frac{\Gamma\left(\nu-\nu'\right)}{\Gamma\left(\nu\right)}\ . \tag{21}$$

Using Bateman's relation (6.5.(7)) (Bateman 1953), we represent the function given by equations (18) and (21) as

$$\tilde{J}_{0,0}\left(\rho\right)=\sqrt{\pi}e^{i\pi\left(\nu+\nu'-1/2\right)/2}e^{-\eta/2}\eta^{\left(\nu-\nu'\right)/2}\Psi\left(\nu,1+\nu-\nu';\eta\right), \tag{22}$$

where $\Psi(\nu,1 + \nu - \nu';\eta)$ is the confluent hypergeometric function,

$$\Psi\left(\nu,1+\nu-\nu';\eta\right)=\frac{\Gamma\left(\nu'-\nu\right)}{\Gamma\left(\nu'\right)}\Phi\left(\nu,1+\nu-\nu';\eta\right)$$
$$+\frac{\Gamma\left(\nu-\nu'\right)}{\Gamma\left(\nu\right)}\eta^{\nu'-\nu}\Phi\left(\nu',1+\nu'-\nu\ ;\eta\right). \tag{23}$$

Using the transformation $\nu \to \nu + j$ and $\nu' \to \nu' + j'$ in equation (22), we obtain the final form

$$\tilde{J}_{j',j}\left(\rho\right)=e^{i\pi\left(\nu+\nu'+j+j'\right)/2}I_{j',j}\left(\rho\right), \tag{24}$$

$$I_{j',j}\left(\rho\right)=\sqrt{\pi}\exp\left[\left(\ln\frac{\rho}{\sqrt{2}}-\frac{i\pi}{4}\right)\left(\nu-\nu'+j-j'\right)+i\frac{\rho^2}{2}-\frac{i\pi}{4}\right]$$
$$\times\Psi\left(\nu+j,1+\nu-\nu'+j-j';-i\frac{\rho^2}{2}\right). \tag{25}$$

The integral $J_{j',j}(\rho)$ given by equation (15) can be represented as the solution of equation (17) where $\Lambda' = \lambda' + i\,(1 - 2j')$ and $\Lambda = \lambda + i\,(2j - 1)$. Using Bateman's relation (8.2.(6)) (Bateman 1953), we transform one of the WPCFs in equation (15) to obtain convenient representations:

$$J_{j',j}\left(\rho\right)=\frac{\Gamma\left(\nu-j+1\right)}{\sqrt{2\pi}}\left[e^{i\pi\left(\nu-j\right)/2}\tilde{J}_{j',1-j}\left(\rho\right)+e^{-i\pi\left(\nu-j\right)/2}J'_{j',1-j}\left(\rho\right)\right], \tag{26}$$

$$J'_{j',1-j}\left(\rho\right)=\int_{-\infty}^{\infty}D_{-\nu'-j'}\left[-\left(1+i\right)u\right]D_{-\nu+j-1}\left[\left(1+i\right)u\right]e^{i\rho u}du, \tag{27}$$

where $\tilde{J}_{j',1-j}\left(\rho\right)$ is given by equation (24). The integral $J'_{j',1-j}(\rho)$ is represented by function (18) where some coefficients C'_1 and C'_2 can be fixed by the comparison with $\rho \to 0$ limit of the integral $J'_{j',1-j}(\rho)$.

Let us start with $J'_{0,0}(\rho)$, where $\Lambda' = \lambda' + i$ and $\Lambda = \lambda + i$. In this case, function (18) takes the form $C'_1 \eta^{i\beta} + C'_2 \eta^{-i\beta}$ as $\rho \to 0$. Hence all ρ -independent terms of $J'_{0,0}(\rho)$ given by equation (27) can be ignored at $\rho \to 0$ limit and only the oscillation terms of the following integrals

$$G^{+}=\int_{0}^{\infty}g^{+}\left(u\right)e^{i\rho u}du,\ \ F^{-}=\int_{-\infty}^{0}g^{-}\left(u\right)e^{i\rho u}du,\ \ g^{\pm}\left(u\right)=g\left(u\right)\Big|_{u\to\pm\infty},$$
$$g\left(u\right)=D_{-\nu'}\left[-\left(1+i\right)u\right]D_{-\nu}\left[\left(1+i\right)u\right] \tag{28}$$

are essential. Using Bateman's relations (8.2.(7)) and (8.4.(1)) (Bateman 1953), we find:

$$J'_{0,0}\left(\rho\right)=\sqrt{\pi}e^{i\pi\left(\nu'-\nu-1/2\right)/2}\left[e^{i\pi\left(\nu-\nu'\right)/4}\frac{\Gamma\left(\nu-\nu'\right)}{\Gamma\left(\nu\right)}\left(\frac{\rho}{\sqrt{2}}\right)^{\nu'-\nu}\right.$$
$$\left.+e^{-i\pi\left(\nu-\nu'\right)/4}\frac{\Gamma\left(\nu'-\nu\right)}{\Gamma\left(\nu'\right)}\left(\frac{\rho}{\sqrt{2}}\right)^{\nu-\nu'}\ \text{as}\ \rho\to 0\right]. \tag{29}$$

Comparing equations (18) and (29), we obtain:

$$C'_1=\sqrt{\pi}e^{i\pi\left(\nu'-\nu-1/2\right)/2}\frac{\Gamma\left(\nu'-\nu\right)}{\Gamma\left(\nu'\right)},\ C'_2=\sqrt{\pi}e^{i\pi\left(\nu'-\nu-1/2\right)/2}\frac{\Gamma\left(\nu-\nu'\right)}{\Gamma\left(\nu\right)}. \tag{30}$$

Using Bateman's relation (6.5.(7)) (Bateman 1953), the function given by equations (18) and (30) can be represented as:

$$J'_{0,0}(\rho) = \sqrt{\pi} e^{i\pi(\nu' - \nu - 1/2)/2} e^{-\eta/2} \eta^{(\nu - \nu')/2} \Psi(\nu, 1 + \nu - \nu'; \eta). \tag{31}$$

Using the transformations $\nu \to \nu + 1 - j$ and $\nu' \to \nu' + j'$ in equation (31), we obtain the following representation for integral (27):

$$J'_{j',1-j}(\rho) = e^{-i\pi(\nu - \nu' + 1 - j - j')/2} I_{j',1-j}(\rho), \tag{32}$$

where $I_{j',j}(\rho)$ is given by equation (25). Substituting representations (24) and (32) into equation (26), we find the final form:

$$J_{j',j}(\rho) = (-1)^j \sqrt{\frac{2}{\pi}} \Gamma(\nu - j + 1) \exp\left[i\pi(\nu' + j' - 1)/2\right] \sinh\frac{\pi\lambda}{2} I_{j',1-j}(\rho), \tag{33}$$

where Γ is the gamma-function.

Conclusion

It was shown that integral representations for the probabilities of one photon emission due to a constant electric field are presented as Fourier transformations of the product of two WPCFs of some kind. We have expressed such a Fourier transformation by the confluent hypergeometric function. It helps greatly in studies of the probabilities of one photon emission that will be presented in a following publication elsewhere.

Conflict of Interest

The author declares that there is no conflict of interest, either existing or potential.

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