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Interaction of subsystems in nonlinear dynamics problems. Various phases of chaos

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Abstract. A model of two interacting dissipative subsystems described by equations of nonlinear dynamics is considered. Each of the subsystems is a nonlinear oscillator driven by an external periodic field. The numerical calculation shows that chaotic oscillations can occur in this system. Their phase trajectories are described by a chaotic attractor in the limit of large times. It is shown that due to the symmetry of the system, different initial conditions can lead to different chaotic attractors. An analogy is discussed between various strange attractors of this model and different phases of matter in systems with a large number of particles.

Keywords: nonlinear dynamics, strange attractor, chaotic attractor, probability density, chaos, thermodynamic phase

Introduction

One of the characteristic features of systems described by nonlinear dynamics equations is the appearance of chaotic motion. In particular, in dissipative systems, the trajectory of a system in phase space may tend to a set of points called a chaotic (strange) attractor (Grinchenko et al. 2017; Malinetsky, Potapov 2000; Sagdeev et al. 1988; Schuster 1984). Chaos in such systems is interspersed with regular (periodic) movement when some parameters change. The corresponding parameters are called control parameters.

Note that the appearance of chaotic states is caused exactly by the nonlinearity of systems of differential equations and manifests itself in physical systems that are not described by the equations of classical dynamics. Currently, the methods of modern micro- and nanotechnology have made it possible to create objects with unusual electromagnetic properties, the so-called metamaterials (Soukoulis, Wegener 2010; Zheludev 2010). Among them of special interest are: 2D supercrystals of semiconductor quantum dots (Evers et al. 2013) and super crystals synthesized on the basis of aromatic 2D polymers (Liu et al. 2017). To date, the optical properties of these objects, especially nonlinear ones, including chaotic dynamics, are a promising and little-studied avenue of research (Ryzhov et al. 2017; 2021). In these nonlinear dynamical systems, under conditions of a degenerate doublet state, the problem can be reduced (if there are appropriate conservation laws) to the Duffing equation. Nonlinear optical dynamics based on quasi-resonator superradiance of a two-layer system with active walls (doped Λ-emitters) can be reduced to a system of two coupled Duffing oscillators considered in this article.

Chaotic motion in such systems has some features in many ways similar to the features in systems with a large number of particles. In particular, similarly to systems with a large number of particles, the states of chaotic systems in question can be described in terms of probability by defining the concept of probability density. As is in systems with a large number of particles, the probability density determines the probability of finding the system in a given region of phase space (Kuznetsov 2006; Liapzev 2019). For systems with some nontrivial symmetry, the probability density is transformed in accordance with a fully symmetric representation of the corresponding symmetry group (Liaptsev 2013; 2014).

Continuing the analogy between the chaotic state of a dissipative system described by a nonlinear dynamic equation and a system with a large number of particles, it can be noted that a strange attractor is analogous to the equilibrium state of a system with a large number of particles. This means that, regardless of the initial state, over time, systems tend to some equilibrium state described by some probability density function. It is the independence from the initial state that makes it possible to determine the chaotic attractor in problems of nonlinear dynamics. However, referring to systems with a large number of particles, one can find examples when, under the same external conditions, the existence of several equilibrium states with different properties is possible. In such cases, the states correspond to different phases.

A typical example is the states arising during the transition from the gas phase to the liquid phase, described by isotherms of a real gas (Fig. 1).

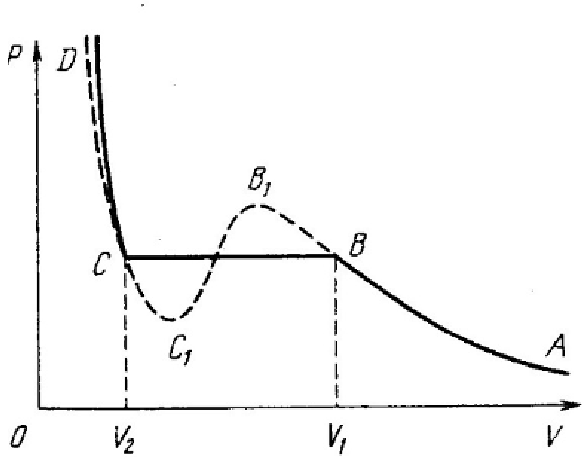

Fig.1. Isotherms of a real gas

In the interval of the isotherm described by the straight *BC*, the state is characterized by the equilibrium of two phases — liquid and gas. However, along with this stable state, states with a single phase are possible in intervals CC_1 and BB_1. This is the so-called "supercooled vapor" in the BB_1 interval and "superheated liquid" in the CC_1 interval. These states are less stable than the states on the *BC* line, and can only be obtained with an adiabatic change in the system parameters from some other states (Kondepudi, Prigogine 2015). In this paper, we consider a model of a dissipative system described by equations of nonlinear dynamics, the state of which may tend to various chaotic attractors depending on the initial states of the system. This is equivalent to the existence of states with different phases under the same conditions in systems with a large number of particles.

Formulation of the model

One of the systems in which chaos is observed is a model of a nonlinear oscillator driven by external periodic force. Such systems are described by the Duffing equation:

$$\ddot{x}(t) + \gamma \dot{x}(t) - \alpha x(t) + \beta x^3(t) = F \sin(\omega t)\,. \tag{1}$$

In this equation, the variable x describes the oscillations, the points above the variable are the standard notation of time derivatives, the parameters α and β determine the shape of the oscillator potential. In particular, at $\alpha > 0$ and $\beta > 0$, the so-called *W*-potential appears. The parameter γ characterizes the value of dissipation, and the parameters F and ω the amplitude and frequency of external action. A real system corresponding to this model can be a load that is located in the upper part of an elastic rod and swings in one plane to an external force (Fig. 2).

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Fig. 2. A real system corresponding to the Duffing equation

Next, we will consider a model of two spring-coupled nonlinear oscillators (Fig. 3).

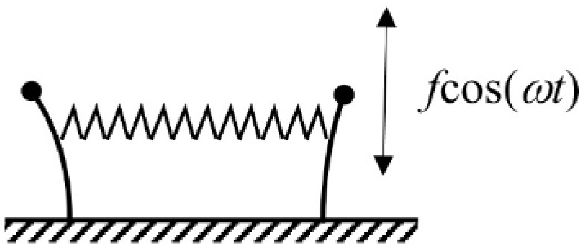

Fig. 3. A model of two oscillators connected by a spring

The stiffness of the spring k characterizes the magnitude of the interaction of subsystems. At $k \to 0$, the interaction of subsystems disappears. We will also assume that an external force acts vertically. In the case of a pendulum, the moment of such force is proportional to sin(x), where x is the angle of deviation from the vertical. To take into account the bending of the rod, we will model the moment of external force proportional to sin(px), where p is some constant. As a result, the system of dynamic equations for the formulated model takes the form:

$$\ddot{x}_1 = f\sin(px_1)\sin(\omega t) - \gamma\dot{x}_1 + \alpha x_1 - \beta x_1{}^3 - k(x_1 - x_2),$$
$$\ddot{x}_2 = f\sin(px_2)\sin(\omega t) - \gamma\dot{x}_2 + \alpha x_2 - \beta x_2{}^3 - k(x_2 - x_1).$$

This system of equations can be written as an autonomous system of 1st order differential equations:

$$\begin{aligned}
&\dot{x}_1 = v_1,\\
&\dot{x}_2 = v_2,\\
&\dot{v}_1 = f\sin(px_1)\sin\varphi + \alpha x_1 - \beta x_1^3 - \gamma v_1 - k(x_1 - x_2),\\
&\dot{v}_2 = f\sin(px_2)\sin\varphi + \alpha x_2 - \beta x_2^3 - \gamma v_2 - k(x_2 - x_1),\\
&\dot{\varphi} = \omega.
\end{aligned} \tag{2}$$

The variables v_1 and v_2 are the velocities corresponding to the coordinates x_1 and x_2. The numerical calculation shows that this system of nonlinear differential equations has chaotic solutions for some parameter values. The set of points in the 5-dimensional phase space, to which the trajectory of this solution tends at large times, represents a chaotic attractor. In accordance with what was said above, the state in phase space can be characterized by a probability density $\rho(x_1, x_2, v_1, v_2, \varphi)$ determined in such a way that the probability of finding a system in a small volume of phase space $\Delta x_1 \Delta x_2 \Delta v_1 \Delta v_2 \Delta\varphi$ is equal to:

$$\Delta w = \rho(x_1, x_2, v_1, v_2, \varphi)\Delta x_1 \Delta x_2 \Delta v_1 \Delta v_2 \Delta\varphi . \quad (3)$$

The probability density satisfies a partial differential equation of the 1st order (Liapzev 2019) (see also (Kuznetsov 2006)), which takes the following form for the system of equations (2):

$$\begin{gathered} \omega\frac{\partial\rho}{\partial\varphi} + v_1\frac{\partial\rho}{\partial x_1} + v_2\frac{\partial\rho}{\partial x_2} + \\ \left(f\sin\varphi\sin(px_1) + \alpha x_1 - \beta x_1^3 - \gamma v_1 - k(x_1 - x_2)\right)\frac{\partial\rho}{\partial v_1} + \\ \left(f\sin\varphi\sin(px_2) + \alpha x_2 - \beta x_2^3 - \gamma v_2 - k(x_2 - x_1)\right)\frac{\partial\rho}{\partial v_2} = 0 \end{gathered} . \quad (4)$$

Analysis of particular solutions

Note, first of all, that the system of equations (2) has a trivial solution $x_1 = x_2 = v_1 = v_2 = 0,\ \varphi = \omega t$, which is realized under initial conditions $x_1(0) = x_2(0) = v_1(0) = v_2(0) = \varphi(0) = 0$. In the special case *k*=0, depending on the initial conditions, one of the subsystems may have an identically zero solution, and the second may have some non-zero solution, including a chaotic solution. These solutions, the initial conditions under which they are implemented, as well as the form of the solution for the probability density of the state in the case when one of the solutions is chaotic, are given in Table 1. In all cases $\varphi(t) = \omega t$.

Table 1. Probability density depending on initial conditions

№	Solutions	Initial conditions	Probability density of the state $\rho(x_1, x_2, v_1, v_2, \varphi)$
1	$x_2(t) = v_2(t) = 0,$ $x_1(t) \neq 0,\quad v_1(t) \neq 0$	$x_2(0) = v_2(0) = 0$	$\rho(x_1, v_1, \varphi)\delta(x_2)\delta(v_2)$
2	$x_1(t) = v_1(t) = 0,$ $x_2(t) \neq 0,\quad v_2(t) \neq 0$	$x_1(0) = v_1(0) = 0$	$\rho(x_2, v_2, \varphi)\delta(x_1)\delta(v_1)$
3	$x_1(t) = x_2(t),$ $v_1(t) = v_2(t)$	$x_1(0) = x_2(0),$ $v_1(0) = v_2(0)$	$\rho(x_1, v_1, \varphi)\delta(x_1 - x_2)\delta(v_1 - v_2)$
4	$x_1(t) = -x_2(t),$ $v_1(t) = -v_2(t)$	$x_1(0) = -x_2(0),$ $v_1(0) = -v_2(0)$	$\rho(x_1, v_1, \varphi)\delta(x_1 + x_2)\delta(v_1 + v_2)$

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The right column of the table shows the standard designation for δ-function. The numerical calculation under the initial conditions given in the table confirms the existence of chaotic solutions satisfying the relations given in the table.

Symmetry properties of equations and solutions

A simple analysis allows us to determine the symmetry properties of the system of equations (2). Namely, the system of equations remains invariant with simultaneous replacement of indices and inversion of variables x_i and v_i. For linear systems of equations, certain relations for solutions of this system follow from such symmetry. In this case, the symmetry properties also allow us to obtain some properties of solutions. To obtain these properties, we introduce symmetric x_s and antisymmetric x_a coordinates and corresponding velocities:

$$x_s = \frac{x_1 - x_2}{2}, \quad x_a = \frac{x_1 + x_2}{2}, \quad v_s = \frac{v_1 - v_2}{2}, \quad v_a = \frac{v_1 + v_2}{2}. \tag{5}$$

The above symmetry transformations leave symmetric variables invariant and change the sign of antisymmetric variables.

The system of equations (2) for the new variables takes the form:

$$\begin{aligned}
&\dot{x}_a = v_a, \\
&\dot{x}_s = v_s, \\
&\dot{v}_a = f\sin(px_a)\cos(px_s)\sin\varphi + \alpha x_a - \beta x_a^3 - \gamma v_a, \\
&\dot{v}_s = f\sin(px_s)\cos(px_2)\sin\varphi + \alpha x_s - \beta x_s^3 - \gamma v_s - 2kx_s, \\
&\dot{\varphi} = \omega.
\end{aligned} \tag{6}$$

Obviously, in the special case k=0, solutions 3 and 4 given in Table 1 correspond to zero symmetric and antisymmetric solutions. However, from the analysis of the system of equations (6), more general conclusions that are valid for all values of parameter k can be deduced. Namely, under initial conditions $x_s(0) = v_s(0) = 0$, solutions are possible in which only antisymmetric variables are different from zero $x_a \neq 0,\ v_a \neq 0$. The obtained solutions, among which there can be both regular and chaotic solutions, do not depend on the parameter k. On the contrary, under initial conditions $x_a(0) = v_a(0) = 0$, solutions are possible in which only symmetric variables are different from zero $x_s \neq 0,\ v_s \neq 0$. Note that the above symmetry transformation and the identity transformation form a symmetry group isomorphic to the point symmetry group C_2. The probability density $\rho(x_1, x_2, v_1, v_2, \varphi)$ for any of the above solutions is transformed by the fully symmetric representation of this group. This means that equality $\rho(-x_2, -x_1, -v_2, -v_1, \varphi) = \rho(x_1, x_2, v_1, v_2, \varphi)$ or equality $\rho(-x_a, x_s, -v_a, v_s, \varphi) = \rho(x_a, x_s, v_a, v_s, \varphi)$ for the variables x_a, x_s, v_a, v_s is fulfilled.

Analysis of numerical solutions

Numerical calculations were carried out as follows. At the first step, the system of differential equations was numerically solved under given initial conditions over a time interval from 0 to $10T$, where T is the period of the driving force. At each of the periods, the values of the variables were calculated in increments of $T/360$. The final state was the initial state for the next interval of 10 periods of the driving force. Then the actions were repeated N times. The number N could be varied. The calculations below correspond to the values of N = 1000, that is, the full-time interval corresponded to 10,000 periods of the driving force.

Under arbitrary initial conditions, both symmetric and antisymmetric oscillations were excited. The calculations were carried out starting from the parameter k = 0 followed by a further increase of this parameter and preservation of the remaining parameters. The calculations have shown that with an increase in the value of *k*, the amplitude of symmetric oscillations decreased on average, and the amplitude of antisymmetric oscillations increased. To quantify this effect, the average volumes of phase space occupied by symmetric and antisymmetric oscillations were calculated. These volumes were defined by calculating the following averages:

$$V_s = \left\langle x_{si}^2 + v_{si}^2 \right\rangle, \quad V_a = \left\langle x_{ai}^2 + v_{ai}^2 \right\rangle, \tag{7}$$

where the index *i* numbers the values of variables in the arrays obtained during the calculations.

Fig. 4 shows graphs of the dependence of *Vs* and *Va* values under arbitrary initial conditions, denoted as Vs_{rand} and Va_{rand}. When constructing the graphs, the results obtained with the values of the variable *k* for regular (periodic in time) solutions were not taken into account.

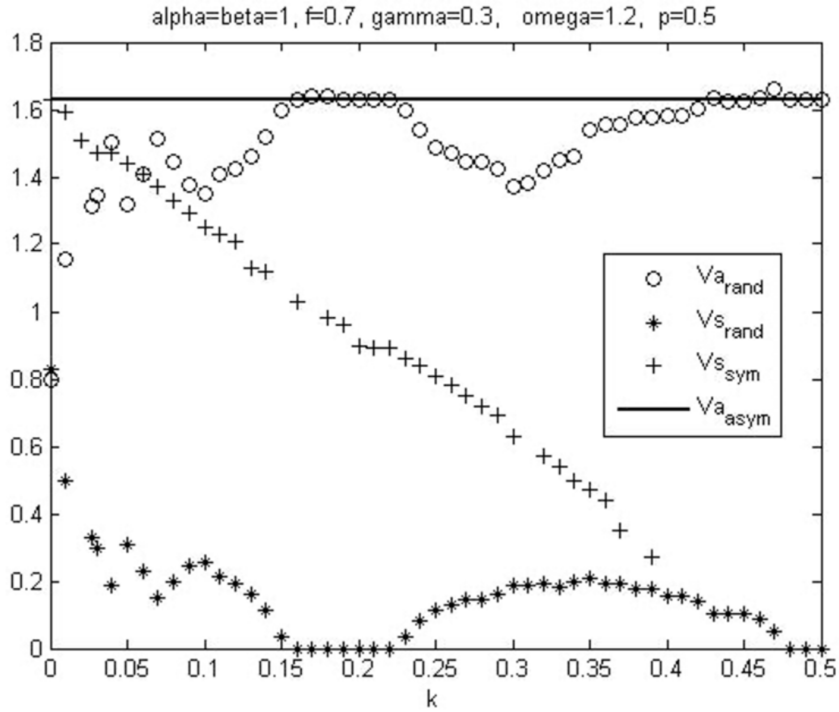

Fig. 4. Results of the numerical calculation

As can be seen from the graphs, the values of $Va_{rand}(0)$ and $Vs_{rand}(0)$ (no interaction of subsystems) are approximately the same. As the value of *k* increases (the spring stiffness increases), the values of $Va_{rand}(k)$ increase, and the values of $Vs_{rand}(k)$ decrease, although the corresponding dependencies are not monotonic. At the same time, the sum of $Va_{rand}(k) + Vs_{rand}(k)$ remains approximately constant. At values $k \geq 0.8$, the oscillations become completely antisymmetric: $Vs_{rand}(k) = 0$. This corresponds to the fact that the subsystems are rigidly connected and move as a whole. The solutions $Va_{rand}(k)$ and $Vs_{rand}(k)$ are stable, that is, under different initial conditions, the obtained values coincide. It should also be noted that there is a range of values $k \in [0.16, 0.22]$ for which $Vs_{rand}(k)$ turns to zero, that is, the oscillations become asymmetric.

A qualitatively different nature of solutions is obtained if the initial states correspond to a symmetric ($x_a(0) = v_a(0) = 0$) or an antisymmetric ($x_s(0) = v_s(0) = 0$) state. For the antisymmetric initial state, the numerical solutions obtained do not depend on the values of the parameter *k*, which corresponds to the above qualitative analysis. This means that $Va(k) = const$, $Vs(k) = 0$. The corresponding graph $Va(k)$ is shown in Fig. 4 as a straight line (Va_{asym}). For the symmetric initial state, the numerical calculation gives the result $Va(k) = 0$ and monotonically decreasing $Vs(k)$ dependencies, shown in Fig. 4 in the notation Vs_{sym}. In the range of values $k \in [0.39, 0.48]$, the solutions of the system of differential equations become regular with the amplitude decreasing with an increase in the value of *k*. For even larger values of *k*, the solutions become chaotic with a small amplitude, so that $Vs(k) \approx 0$.

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Unlike the solutions obtained under arbitrary initial conditions, symmetric (Vs_{sym}) and antisymmetric (Va_{asym}) solutions are unstable. This instability can be detected as follows. As mentioned above, the calculation was carried out in such a way that the final state obtained by solving a system of equations at a certain time interval is the initial state at the next time interval. If the final state is changed to the value of machine precision and taken as the initial state at the next stage, then with further calculation, the state of the system loses its initial symmetry. That is, if at some value of *k* the system was in a state where only a symmetric oscillation was excited (the Vs_{sym} branch in Fig. 4), or only an antisymmetric oscillation (Va_{asym} branch in Fig. 4), then the system jumps into a state in which $Va \neq 0$, $Vs \neq 0$ (Va_{rand} and Vs_{rand} branches in Fig. 4). Such a transition is equivalent to the transition between the states of a real gas, when instead of an overheated liquid or supercooled gas, a state in the form of a mixture of liquid and gas phases occurs.

In systems described by equations of nonlinear dynamics, another important property, similar to what occurs in systems with a large number of particles, is manifested. Namely, with an adiabatic change of parameters, some properties of the system are described by various dependencies with decreasing and increasing parameters. In systems with a large number of particles, the corresponding property is called “hysteresis”. The simplest examples of the manifestation of hysteresis in nonlinear dynamics problems can be used in the educational process (task 2.9 from the textbook (Kondratyev, Liaptsev 2008)). In the model considered here, the phenomenon of hysteresis manifests itself with an adiabatic change in the parameter *k*. The adiabatic change of the parameter means that when the calculation is completed with some parameter *k*, the resulting final state is used as the initial state for the calculation with the parameter $k + \Delta k$. An example of hysteresis in the system considered in our work are the calculations whose results are shown in Fig. 4. In particular, if we start the calculation with the value $k = 0.1$ under unsymmetric initial conditions, when both symmetric and antisymmetric oscillations ($V_a \neq 0, V_s \neq 0$) are excited, and then adiabatically increase the parameter *k*, as described above, we can come to a region $k \in [0.16,\ 0.22]$, where only the antisymmetric oscillation is different from zero, that is, $Vs = 0$. If the parameter *k* is then reduced adiabatically, the state of the system will change so that, as before, the symmetric oscillation turns out to be unexcited. The diagram illustrating the above explanation is shown in Fig. 5.

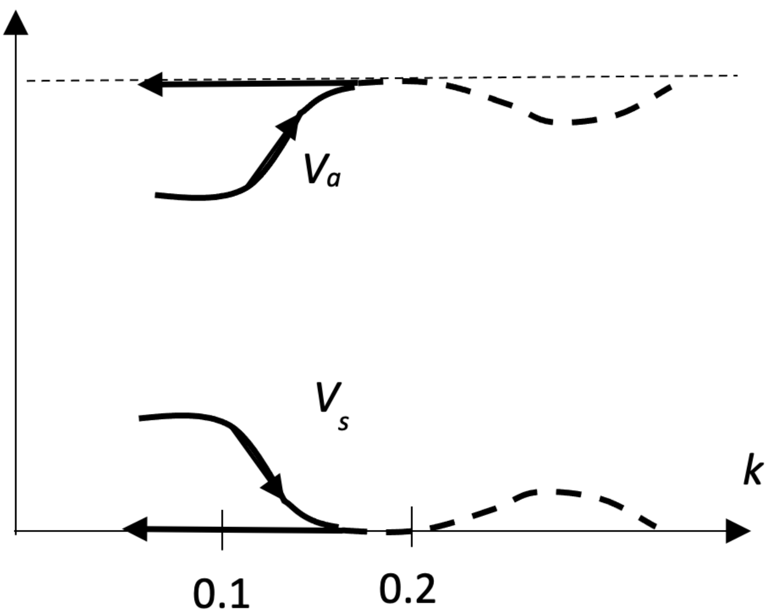

Fig. 5. A diagram illustrating the manifestation of hysteresis with an adiabatic change in the parameter *k*

Similar manifestations of hysteresis are processes on the isotherm of a real gas (Fig. 1). If, at the initial state (a mixture of gas and liquid on a rectilinear section of the *CB* isotherm), the volume is increased, then at point *B* the system is in a homogeneous gas state. If you then slowly reduce the volume, you can get superheated steam, that is, start moving along the section of the BB_1 isotherm.

Conclusions

Returning to the discussion of the analogy of the chaotic solutions obtained with the state of a real gas, the following analogies can be stated.

1) For an arbitrary initial state of the system under consideration, the solution corresponding to chaotic oscillations is a mixture of symmetric and antisymmetric oscillations. This is analogous to the fact that the state of the real gas in the *BC* isotherm region (Fig. 1) is a mixture of liquid and gaseous phases.
2) When the parameter *k* is changed, the contribution of symmetric and antisymmetric oscillations changes up to the moment when the contribution of symmetric oscillations becomes zero. This is analogous to the fact that when the volume of a real gas changes, the proportion of liquid and gas changes, too, and only one of the phases remains at points B or C (Fig. 1).
3) The "coexistence" of phases, that is, the state in which $Va \neq 0$, $Vs \neq 0$, is stable, similarly to the stable state of the real gas at the *BC* region.
4) In addition to the stable state as a "mixture of phases" ($Va \neq 0$, $Vs \neq 0$), there are unstable states formed under specific initial conditions (symmetric or antisymmetric initial state). An analog for the isotherm of a real gas is the presence of states with overheated liquid or supercooled gas.

These analogies indicate the possibility of the existence of various phases of a chaotic state in systems described by equations of nonlinear dynamics.

Conflict of Interest

The author declares that there is no conflict of interest, either existing or potential.

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