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Does a primary hair have an impact on the naked singularity formation in hairy Vaidya spacetime?

V. D. Vertogradov ✉[1,2]

[1] Herzen State Pedagogical University of Russia, 48 Moika Emb., Saint Petersburg 191186, Russia
[2] SPb branch of SAO RAS, 65 Pulkovskoe Highway, Saint Petersburg 196140, Russia

Authors
Vitalii D. Vertogradov, ORCID: 0000-0002-5096-7696, e-mail: vdvertogradov@gmail.com

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Abstract. We consider the gravitational collapse of Vaidya spacetime, which has been obtained using the gravitational decoupling method. In this paper, we are interested in whether a primary hair has any impact on the endstate of the gravitational collapse. We also prove that the coupling constant has an influence on the naked singularity formation. The strength of the central singularity has also been investigated, and we show that the naked singularity is gravitationally strong. However, this model does not violate the cosmic censorship conjecture because in the case of the naked singularity formation the weak energy condition is violated.

Keywords: black hole, naked singularity, gravitational collapse, gravitational decoupling, strength of singularity

Introduction

When a star exhausts all its fuel, it undergoes continuous gravitational collapse. Oppenheimer and Snyder (Oppenheimer, Snyder 1939) were one of the first who considered the model of homogeneous dust collapse leading to a black hole formation. According to the cosmic censorship conjecture [CCC], a singularity must be covered by the horizon. However, if one considers the gravitational collapse of inhomogeneous dust (Jhingan et al. 1996; Singh, Joshi 1996), then the result might be a singularity that is not covered by the apparent horizon — so-called naked singularity. Thorough investigation of different scenarios of the gravitational collapse showed that under physically relevant conditions, a naked singularity might form during this process (Dey et al. 2022; Goncalves, Jhingan 2001; Harko 2003; Joshi 2007; Joshi, Malafarina 2011; Mosani et al. 2022; Naidu al. 2020).

Papapetrou (Papapetrou 1985) showed that the Vaidya spacetime (Vaidya 1951) is an example of the CCC violation. The radiating Schwarzschild spacetime, or Vaidya metric, is widely used in many astrophysical applications with strong gravitational fields. The spacetime around a black hole should be dynamical due to the processes of emission and absorption, and the Vaidya spacetime is one of the simplest examples of a dynamical spacetime. It is used to describe the exterior geometry of radiating stars (Santos 1985). The backreaction of the accreting matter leads to the Vaidya spacetime in spherically-symmetric case (Babichev et al. 2012). The question about the dynamical shadow formation in Vaidya and charged Vaidya spacetimes is discussed in (Heydarzade, Vertogradov 2023; Koga et al. 2022; Solanki, Perlick 2022). The horizon structure and entropy of this solution are investigated for an empty background in (Nielsen 2014; Nielsen, Yoon 2008; Vertogradov, Kudryavcev 2023), and for Vaidya surrounded by cosmological fields in (Heydarzade, Darabi 2018a; 2018b; 2018c). The Vaidya spacetime can be extended by including the null strings, leading to the generalized Vaidya one (Wang, Wu 1999). The gravitational collapse and naked singularity formation in this spacetime have been discussed in (Acquaviva et al. 2015; Dey, Joshi 2019; Mkenyeleye et al. 2014; Ray et al. 2022; Vertogradov 2016; 2018; 2022). The generalized Vaidya spacetime has an off-diagonal term, which may lead to negative energy states for a particle like in Kerr spacetime (Penrose, Floyd 1971). The absence of such particles has been proven in (Vertogradov 2020). However, there is a similar effect for the charged particles, and the generalized Penrose process may take place in static Reissner–Nordström (Denardo, Ruffini 1973) and dynamical charged Vaidya (Vertogradov 2023) cases.

The method of extended gravitational decoupling (Ovalle 2017) allows to obtain new solutions to the Einstein field equations with a more realistic energy–momentum tensor. A generalization of Schwarzschild, Kerr and Hayward black holes has been obtained in papers (Mahapatra, Banerjee 2023; Ovalle et al. 2021; Vertogradov, Misyura 2023). Radiating hairy Schwarzschild black holes have been obtained in (Vertogradov, Misyura 2022). The hairy Vaidya spacetime can be considered as a model of gravitational collapse with a new parameter of length dimension, which can be regarded as a primary hair. In this paper, we consider the model of the gravitational collapse of the hairy Vaidya solution in order to find out if a primary hair can affect the endstate of the gravitational collapse. We also consider the strength of the singularity.

This paper is organized as follows: in Section 2 we briefly describe the Vaidya spacetime obtained by the extended gravitational decoupling method. In Section 3 we consider the gravitational collapse model and find out if a primary hair can affect the endstate of such a process. In Section 4 we investigate the strength of the central singularity, and Section 5 is the conclusion.

Throughout this paper, we use the system of geometrized units, in which $G = 1 = c$. Also, we use the signature $-+++$.

Vaidya spacetime by gravitational decoupling

The famous no-hair theorem states that a black hole might have only three charges: mass M, angular momentum J, and electric charges Q (Ruffini, Wheeler 1971). However, it can be shown that black holes can have other charges, and there is so-called soft hair (Hawking et al. 2016). One of the possibilities of evading the no-hair theorem is to use the gravitational decoupling method (Contreras et al. 2021; Ovalle 2017; 2019).

It is well known that obtaining the analytical solution of Einstein equations is a difficult task in most cases. We know that we can obtain an analytical solution of the spherically symmetric spacetime in the case of the perfect fluid as the gravitational source. However, if we consider the more realistic case when the perfect fluid is coupled to another matter, it is nearly impossible to obtain the analytical solution. In papers (Contreras et al. 2021; Ovalle 2017; 2019;), it was shown using the Minimal Geometric Deformation (MGD) (Babichev, Charmousis 2014; Sotiriou, Faraoni 2012) method that we can decouple the gravitational sources; for example, one can write the energy–momentum tensor T_{ik} as:

$$T_{ik} = \tilde{T}_{ik} + \alpha \Theta_{ik} \quad , \tag{1}$$

where $\tilde{T}_{ik}$ is the energy–momentum tensor of the perfect fluid and α is the coupling constant to the energy–momentum tensor Θ_{ik}. It is possible to solve Einstein's field equations for a gravitational source whose energy–momentum tensor is expressed as (1) by solving Einstein's field equations for each component $\tilde{T}_{ik}$ and Θ_{ik} separately. Then, by a straightforward superposition of the two solutions, we obtain the complete solution corresponding to the source T_{ik}. Since Einstein's field equations are non-linear, the MGD decoupling represents a novel and useful method in the search for and analysis of solutions, especially when we face scenarios beyond trivial cases, such as the interior of stellar systems with gravitational sources being more realistic than the ideal perfect fluid, or even when we consider alternative theories, which usually introduce new features that are difficult to deal with.

Moreover, there is only the gravitational interaction between two sources, i. e.

$$T^{ik}_{;k} = 0 \rightarrow \tilde{T}^{ik}_{;k} = \alpha \Theta^{ik}_{;k} = 0 \quad . \tag{2}$$

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This fact allows us to think about Θ_{ik} as dark matter. By applying the gravitational decoupling method, one can obtain well-known black hole solutions with hair (Heydarzade et al. 2023; Ovalle et al. 2018; 2021). By using this method, the Vaidya spacetime has been obtained (Vertogradov, Misyura 2022), which in Eddington–Finkelstein coordinates has the following form:

$$ds^2 = -\left(1 - \frac{2m(v)}{r} + \alpha e^{-\frac{r}{M(v)}}\right) dv^2 + 2dvdr + r^2 d\Omega^2 \quad , \tag{3}$$

where $M(v)$ is the mass function which corresponds to usual Vaidya spacetime with $\alpha \equiv 0$. A new mass function $m(v)$ is related to the Vaidya one by

$$m(v) = M(v) + \frac{\alpha l}{2} \quad , \tag{4}$$

where $\alpha > 0$ and $l > 0$ are primary hairs. The energy–momentum tensor supporting the solution (3) is

$$\begin{aligned}
\Theta_0^0 &= \Theta_1^1 = \frac{e^{-\frac{r}{M(v)}}(M(v)-r)}{M(v)r^2} \quad , \\
\Theta_2^2 &= \Theta_3^3 = \frac{e^{-\frac{r}{M(v)}}(-2M(v)+r)}{2rM(v)^2} \quad , \\
\Theta_0^1 &= -\frac{\dot{M}(v)e^{-\frac{r}{M(v)}}}{r^2 M^2(v)} \quad , \\
\tilde{T}_0^1 &= \frac{2\dot{M}(v)}{r^2} \quad .
\end{aligned} \tag{5}$$

Note that we are interested in gravitational collapse, so we assume that $\dot{M} > 0$. The energy–momentum component of the usual Vaidya spacetime $\tilde{T}_0^1 > 0$ has positive value, i. e., positive energy flux. On the other hand, the component $\Theta_0^1 < 0$ which corresponds to the negative energy flux. The coupling constant α is supposed to be very small $\alpha << 1$. Thus, the negative energy flux of the extra matter field can be interpreted as the Hawking radiation (Hawking 1975).

Gravitational collapse model

Consider the vector field K^i tangent to a non-spacelike family of geodesics ($K^i = \frac{dx^i}{d\lambda}$, where λ is an affine parameter). The fact that K^i is tangent to the geodesic congruence and parallel-transported along them gives us the following equation:

$$K^i_{;j} K^j = 0 \quad , \tag{6}$$

and

$$g_{ij} K^i K^j = \delta \quad , \tag{7}$$

where $\delta = 0$ for null and $\delta = -1$ for timelike geodesics.

From geodesic equations for K^2 and K^3 one has

$$\begin{aligned}
K^2 &= \frac{L}{r^2 \sin\beta \cos\varphi} \quad , \\
K^3 &= \frac{L\cos\beta}{r^2 \sin^2\theta} \quad ,
\end{aligned} \tag{8}$$

where L and β are constants of integration, L is an impact parameter, and β is the isotropy parameter such that $\sin\phi\tan\beta = \cot\theta$.

Let us write K^0 as

$$K^0 = \frac{Q(v,r)}{r} \quad . \tag{9}$$

Here, Q is an arbitrary function of both v and r. By using the condition $g_{ik}K^iK^k = \delta$, one obtains

$$K^1 = \frac{Q}{2r}\left(1 - \frac{2m(v)}{r} + \alpha e^{-\frac{r}{M}}\right) - \frac{l^2}{2rQ} + \frac{\delta r}{2Q} \quad . \tag{10}$$

Combining (9) and (10), one gets

$$\frac{dK^0}{d\lambda} = \frac{d}{d\lambda}\left(\frac{Q}{r}\right) = \frac{1}{r}\frac{dQ}{d\lambda} - \frac{Q}{r^2}\frac{dr}{d\lambda} \quad . \tag{11}$$

The geodesic equation, together with (10), gives us the equation with respect to $\frac{dQ}{d\lambda}$:

$$\frac{dQ}{d\lambda} = \frac{Q^2}{2r^2}\left[1 - \frac{4m}{r} + \alpha\left(\frac{7}{2} - \frac{r}{2M}\right)e^{-\frac{r}{M}}\right] + \frac{l^2}{2r^2} + \frac{\delta}{2} \quad . \tag{12}$$

If we know the mass function $m(v)$ and initial conditions, then we can find the solution of this equation (Dwivedi, Joshi 1989).

A naked singularity can be the result of a continuous gravitational collapse if one can satisfy the following two conditions:

1. The time of the singularity formation is less than the time of the apparent horizon formation;
2. There is a family of non-spacelike future-directed geodesics that terminate in the central singularity in the past;

The apparent horizon equation in the case of metric (3) is given by (Vertogradov, Misyura 2022):

$$g_{00} = 0 \rightarrow 1 - \frac{2m(v)}{r} + \alpha e^{-\frac{r}{M(v)}} = 0 \quad . \tag{13}$$

The approximate location of the apparent horizon is

$$r_{ah} = 2M(v) - \alpha(l - 2M(v)e^{-2}) \quad . \tag{14}$$

To prove the existence of the outgoing geodesics that terminate in the past in the singularity, we consider the null radial geodesic equation, which, for the metric (3), has the following form:

$$\frac{dv}{dr} = \frac{2}{1 - \frac{2m(v)}{r} + e^{-\frac{r}{M(v)}}} \quad . \tag{15}$$

The condition for the naked singularity formation is $m(0) = 0$ (Joshi 2007; Mkenyeleye et al. 2014). Let us consider the following limits:

$$\lim_{v\to 0,\, r\to 0} \frac{v}{r} = X_0 \quad ,$$
$$\lim_{v\to 0} \frac{dm(v)}{dv} = \dot{M}_0 \quad . \tag{16}$$

If $\frac{dv}{dr}$ is positive and finite at the singularity, then there exists a family of non-spacelike geodesics, and the result of the gravitational collapse will be a naked singularity. Note that we demand a finite value of $\frac{dv}{dr}$ in order to have

$$\lim_{v\to 0,\, r\to 0} g_{00} \neq 0 \quad . \tag{17}$$

Applying the notations (16) in (15), one obtains

$$X_0 = \frac{2}{1 - 2\dot{m}_0 X_0 + \alpha} \quad . \tag{18}$$

From the metric expression (3) one can see that if $m(0) = 0$, then automatically, by applying (4), $M(0) = -\frac{\alpha l}{2} \neq 0$. So, if one finds the finite value of X_0 by demanding $\alpha \neq -1$, then one will find that at the time $v = 0$ of the singularity formation the apparent horizon is absent. So, let us solve an alge-

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braic equation (18) to find out if the finite and positive values of X_0 are possible. The solution of this equation is:

$$X_0^{\pm} = \frac{1+\alpha \pm \sqrt{(1+\alpha)^2 - 16\dot{m}_0}}{4\dot{m}_0} \quad . \tag{19}$$

From (19) one can easily see that one should satisfy the following condition to have finite and positive values of X_0:

$$\dot{m}_0 \leq \frac{1+\alpha}{16} \quad . \tag{20}$$

If the condition (20) is satisfied, then the result of the gravitational collapse is a naked singularity formation. If we consider the linear mass function

$$m(v) = \mu v \quad , \mu > 0 \quad , \tag{21}$$

then from (20) one obtains the following conditions:

$$\mu \leq \frac{1+\alpha}{16} \quad . \tag{22}$$

The comparison with usual Vaidya spacetime ($\mu \leq \frac{1}{16}$) shows that if $\alpha > 0$, then additional matter field might lead to the naked singularity formation when in usual Vaidya one has a black hole formation. For example, if one considers $\mu = \frac{1}{16} + \frac{\alpha}{8} > \frac{1}{16}$, then in Vaidya spacetime the result of the gravitational collapse is a black hole, but in hairy Vaidya, it is a naked singularity. One has the opposite situation in the case of $\alpha < 0$.

The strength of the central singularity

If we follow Tipler definition (Tipler 1977) which was given in the paper (Nolan 1999), a singularity is termed to be gravitationally strong or simply strong if it destroys by stretching or crushing any object which falls into it. If it does not destroy any object this way, then the singularity is termed to be gravitationally weak. We will consider the radial null geodesic with tangent vector K^i that terminates in the central singularity in the past, i. e., $v = r = \lambda = 0$. Following Clarke and Krolack (Clarke, Królak 1985), a singularity would be strong if the condition

$$\lim_{\lambda \to 0} \lambda^2 R_{ik} K^i K^k > 0 \quad . \tag{23}$$

Here R_{ik} is the Ricci tensor, which for the metric (3) is given by:

$$R_{01} = \frac{\alpha e^{-\frac{r}{M}}}{rM} - \frac{\alpha e^{-\frac{r}{M}}}{2M^2} \quad ,$$
$$R_{00} = -\left(1 - \frac{2M}{r} + \alpha e^{-\frac{r}{M}}\right) R_{01} + \frac{1}{r^2}\left(2\dot{M} + \frac{\alpha r^2 \dot{M}}{M^2} e^{-\frac{r}{M}}\right) \quad . \tag{24}$$

The R_{00} component can be rewritten as

$$R_{00} = -2\frac{dr}{dv} r_{01} + \frac{1}{r^2}\left(2\dot{m} + \frac{\alpha r^2 \dot{m}}{M^2} e^{-\frac{r}{M}}\right) \quad . \tag{25}$$

By combining (24), (22) and (23), we obtain

$$\lim_{\lambda \to 0} \lambda^2 R_{ik} K^i K^k = 2\dot{m}_0 X_0^2 \quad . \tag{26}$$

If the condition for the naked singularity formation is satisfied, then $X_0 > 0$ is finite. So, for the singularity to be strong, one should demand $\dot{m}_0 \neq 0$. Also, one should see that the condition for the singularity to be strong, in the linear mass function case, does not depend upon a primary hair[1]. So for the

[1] In fact, as one can see from (4), the mass function m depends upon primary hairs α and l, but in order to satisfy the condition $m(0) = 0$ one should return to the usual Vaidya mass function by linear coordinate transformation.

linear mass function, the singularity is gravitationally strong. Also, the result of the paper (Vertogradov 2022a) is satisfied, i. e. if one considers the mass function in the form $m(v) = \mu v^{\xi}$, $\xi > 1$) then, as one can see, the condition (26) is not satisfied.

Conclusion

In this paper, the gravitational collapse of hairy Vaidya spacetime has been considered. The result of such a process can lead to the naked singularity formation. A coupling constant α has an impact on the endstate of the gravitational collapse. In usual Vaidya spacetime $\alpha = 0$ the result might be a black hole formation if in the case of the linear mass function $M(v) = \mu v$ the condition $\mu > \frac{1}{16}$ is held. However, α strengthens these conditions, and the endstate of the gravitational collapse might be the naked singularity while, in usual Vaidya, the same conditions would lead to a black hole formation. We also noted that a primary hair l does not have any impact on the endstate of the continuous gravitational collapse.

We have also considered the strength of the central singularity and proved that the singularity is gravitationally strong. It means that this model violates the CCC. However, we should also note that the extra field Θ_{ik} leads to the negative energy flux, which might be interpreted as a Hawking radiation. However, the combination

$$T_0^1 = \tilde{T}_0^1 + \alpha\Theta_0^1 \ , \tag{27}$$

might be positive for the black hole. This problem is analogous to the weak energy condition violation near the central singularity in charged Vaidya spacetime (Ori 1991). So the answer to the question in the title of this paper is the following: a primary hair l does not have any impact on the naked singularity formation, but the coupling constant α does. However, this model violates energy conditions and does not violate the cosmic censorship conjecture. The spacetime (3) should be used to describe the exterior geometry of radiating or accreting massive black holes when all energy conditions are satisfied.

Conflict of Interest

The author declares that there is no conflict of interest, either existing or potential.

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