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# Quasienergy of chaotic states in problems of nonlinear dynamics. Degeneration of states due to the symmetry of the system

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***Abstract.*** In this paper we analyse chaotic states in a system of coupled Duffing oscillators. The concept of quasienergy of a system is introduced in a way similar to the concept of quasienergy of a quantum mechanical system driven by an external periodic field. We show that in the absence of a connection between the oscillators in the system under consideration, chaotic states with the same value of quasienergy, but different values of the angular momentum are realized when the external influence changes. This fact can be interpreted as the existence of degenerate chaotic states of the system. A numerical experiment shows that taking into account the interaction between oscillators leads to the splitting of quasienergy, similar to the splitting of the quasienergy level in a quantum mechanical system.

***Keywords:*** nonlinear dynamics, chaotic states, chaotic attractor, probability density, quasienergy, degenerate states, numerical experiment

## Introduction

Chaotic states in dissipative systems described by equations of nonlinear dynamics have a number of features specific to systems whose states are determined by linear equations. This is explained by the fact that over time, the chaotic states of dissipative systems tend towards a certain set in the phase space called a ‘strange attractor’ or ‘chaotic attractor’ (Loskutov 2007). The state described by the chaotic attractor can be characterized by the probability density, which determines the probability of finding a system in a given region of phase space (Sagdeev et al. 1988). The average values of various quantities characterizing these systems can be calculated using probability density, just as it is done for other systems described by probability density: for example, for systems with a large number of particles or quantum mechanical systems.

The equation for the probability density of a system characterized by a chaotic attractor is a linear equation, similar, for example, to the Schrodinger equation for a wave function, the modulus square of which also determines the probability density (Liaptsev 2019). The linearity of the equation for probability density implies a number of properties characteristic of systems described by linear equations. In particular, the response of the system to small perturbations is small and proportional to the small parameter characterizing the perturbation (Liaptsev 2020). This makes it possible to apply perturbation theory in the same way as it is applied to systems described by the equations of quantum mechanics.

It should be noted, however, that when considering a quantum mechanical system in an external field, it is necessary to consider a more general equation for the density matrix instead of the Schrodinger equation (Blum 2012). With sufficiently strong external fields, the system of equations becomes nonlinear. For external fields of optical frequency, such effects are widely studied in a variety of works on nonlinear optics (Allen, Eberly 1987; Andreev et al. 1993; Bayramdurdiyev et al. 2020; 2021; Benedict et al. 1996; Ryzhov et al. 2016; 2017; Ryzhov et al. 2019; 2021a; 2021b).

Other features of the chaotic states of dissipative systems described by the equations of nonlinear dynamics are properties reflecting the symmetry of such systems. These properties are manifested, for example, in the polarization of radiation from such systems (Liaptsev 2014; 2015). The polarization properties, characterized in particular by the Stokes parameters, are similar to the polarization properties of symmetric quantum mechanical systems in degenerate states. This allows us to make the assumption that the chaotic states of dissipative systems can also be degenerate in a certain sense.

For systems whose description is based on the laws of quantum mechanics, degeneracy is defined as the existence of several states having the same energy. It should be noted, however, that systems whose state tends towards a chaotic attractor are open systems, so that the energy of such systems, if any can be determined, is not conserved over time. However, in most cases, chaotic states in such systems arise when the system is subjected to external periodic influence. These systems include, in particular, such model systems as a nonlinear oscillator and a mathematical pendulum located in an external periodic field (Duffing 1918; Grinchenko et al. 2007; Hacken 1978; Kuznetsov et al. 2002; Moon 1987; Sagdeev et al. 1998). These physical systems have one degree of freedom, and the corresponding equations of nonlinear dynamics in an external periodic field are reduced to a system of 3 differential equations of the 1st order. Therefore, such systems are sometimes called systems with 1.5 degrees of freedom. In the problems considered by quantum theory, when describing systems that are driven by an external periodic field, the concepts of quasienergy and, accordingly, quasienergetic states are used (Bordo et al. 1984; Delone, Krainov 1999; Kiselev, Liapzev 1990; Lyaptsev 1994; Zel'dovich 1973). The time-dependent Schrodinger equation for such systems has the following form:

$$i\hbar \frac{\partial \Psi}{\partial t} = (H_0 + V(t))\Psi \ , \tag{1}$$

where $H_0$ is a Hamiltonian in the absence of an external field, and $V(t)$ is a periodic function of time. According to Bloch's theorem, the solution of this equation can be represented as a superposition of solutions of the form:

$$\Psi(t) = \exp\left(-\frac{iEt}{\hbar}\right)\psi(t) \ , \tag{2}$$

where $\psi(t)$ is a wave function that periodically depends on time with the period of the external field. By definition, the $E$ value is called quasienergy, and $\psi(t)$ is the wave function of a quasienergetic state (QES) (Zel'dovich 1973). As in the case of stationary states, QES can be degenerate, that is, several different wave functions can correspond to one value of quasienergy.

When considering dissipative systems described by equations of nonlinear dynamics, the density matrix corresponding to the chaotic attractor also turns out to be periodically time-dependent. This means that for such systems it is also possible to define the concept of quasienergy, using, for example, the limiting transition from quantum mechanics to classical theory. Below, we will apply a similar approach to describe a model system of coupled Duffing oscillators and show that degenerate chaotic states can occur in this case. As it will be shown, the degeneracy in this case is due to the symmetry of the problem, and with a decrease in symmetry, an effect similar to splitting energy levels with a decrease in symmetry in a quantum mechanical problem may occur.

## Quasienergies of chaotic states of systems driven by an external periodic field

Let us consider, for simplicity's sake, the case of one-dimensional motion of a single particle in a field with potential energy $U(x,t)$, which depends on the coordinates of the particle $x$ and also periodically depends on time. The Hamiltonian included in the Schrodinger equation (1) can be written as:


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$$H = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + U(x,t) \ .$$

Substitution of a solution of the form (2) into the Schrodinger equation leads to the equation for the QES:

$$\frac{\hbar^2}{2m}\psi'' + (E-U)\psi + i\hbar\dot{\psi} = 0 \ .$$

Here and further, the strokes indicate the derivatives of *x*, and the dot above the symbol is the derivative of *t*. The limiting transition to the classical description is carried out by defining the function $\sigma(x,t)$ (Landau, Lifshitz 1977):

$$\psi = \exp\left(\frac{i\sigma}{\hbar}\right) \ .$$

The equation for the function $\sigma(x,t)$ has the following form:

$$\frac{1}{2m}\left(\sigma'\right)^2 - \frac{i\hbar}{2m}\sigma'' + \dot{\sigma} = E - U \ .$$

The transition to the classical description is carried out by the representation of the function $\sigma(x,t)$ in the form of a power expansion of the Planck constant:

$$\sigma = \sigma_0 + \frac{\hbar}{i}\sigma_1 + \left(\frac{\hbar}{i}\right)^2 \sigma_2 + \ldots \ .$$

In zero approximation, we obtain the equation:

$$\frac{1}{2m}\left(\sigma_0'\right)^2 + \dot{\sigma}_0 = E - U \ .$$

This equation coincides with the Hamilton–Jacobi equation for the action function:

$$\frac{1}{2m}\left(S'\right)^2 + U + \dot{S} = 0 \ ,$$

if you put:

$$S = \sigma_0 - Et \ . \qquad (3)$$

According to the periodicity of the function $\psi(x,t)$, the function $\sigma_0(x,t)$ must also be periodic. This is fulfilled within the classical limit if the classical solution $x(t)$ is a periodic function. Indeed, the Lagrangian:

$$L(x,t) = \frac{\dot{x}^2}{2m} - U(x,t)$$

is in this case a periodic function of time. The action is determined by an integral, which, in accordance with expression (3), can be represented as:

$$S = \int_{t_0}^{t} L(t_1)dt_1 = -Et + \sigma_0(t) \ .$$

It can be seen from this expression that in the case of a periodic solution $x(t)$, quasienergy can be defined by the following expression:

$$E = -\frac{1}{T}\int_{t_0}^{t_0+T} L(t_1)dt_1 = -\langle L(t)\rangle\,, \tag{4}$$

where $T$ is the period of the function $U(x,t)$, and the symbol $\langle\ldots\rangle$ indicates the average value of the Lagrangian over the period.

Note that the explicit calculation of quasienergy can be carried out analytically, for example, in the case of a harmonic oscillator with attenuation driven by an external periodic field. The corresponding equation for the oscillator can be written as:

$$\ddot{x} + \gamma\dot{x} + \omega_0^2 x = f\cos(\omega t)\,.$$

The calculation of quasienergy using formula (4) in this case leads to the expression:

$$E = \frac{f^2}{4\left(\omega_0^2 - \omega^2\right)}\,.$$

This expression corresponds to the quasienergy for an atom in a strong electromagnetic field. The corresponding corrections due to the periodic field are called the dynamic Stark effect (Delone, Krainov 1999).

This expression obtained for a periodic solution can be generalized to chaotic solutions of one-dimensional dissipative systems in a periodic field (a Duffing oscillator, a pendulum in a periodic field). The dynamic system of equations for such systems has the form (Grinchenko et al. 2007):

$$\begin{aligned} &\dot{x} = v, \\ &\dot{v} = F(x) - \gamma v + f(x)\cos(\varphi), \\ &\dot{\varphi} = \omega. \end{aligned} \tag{5}$$

In these equations, $F(x)$ is the force acting on the oscillator, $\gamma$ is the dissipation coefficient, $f(x)$ is the amplitude of the external field, depending on $x$, $\omega$ is the frequency of the external field, the variables $x$ and $v$ correspond to the coordinate and velocity, and the variable $\varphi$ is cyclic with a period of $2\pi$. The chaotic state corresponding to the strange attractor can be described using the probability density $\rho(x,v,\varphi)$, which satisfies the partial differential equation (see, for example, (Liaptsev 2020)):

$$\omega\frac{\partial\rho}{\partial\varphi} + v\,\frac{\partial\rho}{\partial x} + (F - \gamma v + f\cos(\varphi))\frac{\partial\rho}{\partial v} = 0\,. \tag{6}$$

It is convenient to represent the three-dimensional phase space of the system under consideration in the form of a torus with the closure of the variable $\varphi$. The probability density determined by equation (6) must be normalized by one. Averaging over the time variable in expression (4) in the presence of a chaotic attractor should be replaced by averaging over the entire phase space:

$$E = -\langle L\rangle = -\int L(x,v,\varphi)\rho(x,v,\varphi)dxdvd\varphi\,. \tag{7}$$

Note that when performing calculations, it is not necessary to calculate the density matrix. An equivalent result can be obtained by calculating the average value of $L$ for each of the time periods, followed by averaging over a large number of periods.

Finally, the expression for quasienergy (7) can be easily generalized to the case of more complex systems, for example, coupled oscillators (Liaptsev 2023). In this case, the averaging is simply carried out over the whole phase space, which has dimension $2n+1/2$, where $n$ is the number of degrees of freedom of the system in question.

## A model of coupled Duffing oscillators

In systems described by quantum theory, the degeneracy of states with a given energy (also with a given quasienergy) can be due to the symmetry of the system. In this case, the symmetry group must

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contain non-commuting transformations (Landau, Lifshitz 1977; Petrashen, Trifonov 2009). One of the simplest of such groups is the symmetry group $C_{3v}$. The simplest model having such symmetry, the solutions of which can be chaotic, is the model of coupled Duffing oscillators. We will consider three symmetrically arranged oscillators with a nonlinear dependence of force on displacement, connected in pairs by an elastic force. The periodic forces acting on each of the oscillators have the same frequency $\omega$, but may differ in phase. The scheme of such a model is shown in Fig. 1.

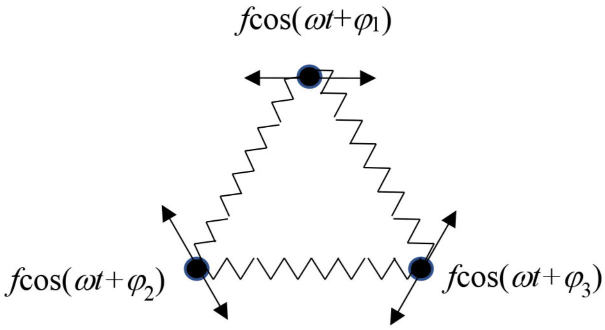


Fig. 1. A model of three coupled Duffing oscillators

The system of dynamic equations for such a model has the form:

$$\begin{aligned}
&\dot{x}_1 = v_1,\\
&\dot{x}_2 = v_2,\\
&\dot{x}_3 = v_3,\\
&\dot{v}_1 = F(x_1) - \gamma v_1 - k(2x_1 - x_2 - x_3) + f\cos(\varphi + \varphi_1),\\
&\dot{v}_2 = F(x_2) - \gamma v_2 - k(2x_2 - x_1 - x_3) + f\cos(\varphi + \varphi_2),\\
&\dot{v}_3 = F(x_3) - \gamma v_3 - k(2x_3 - x_2 - x_1) + f\cos(\varphi + \varphi_3),\\
&\dot{\varphi} = \omega.
\end{aligned} \tag{8}$$

The coefficient $k$ in the equations characterizes the magnitude of the interaction between the oscillators. In the special case, at $k = 0$, there is a system of oscillators not connected by elastic forces.

## Harmonic approximation

The system of equations (8) can be solved in the special case when the force $F(x)$ linearly depends on the displacement: $F(x) = -\omega_0^2 x$. The general solution corresponds to a superposition of forced harmonic oscillations. The normal modes of free undamped oscillations (solution of the system of equations (8) at $f = \gamma = 0$) can be classified by irreducible representations of the symmetry group $C_{3v}$. The displacements corresponding to these fluctuations are shown in Fig. 2.

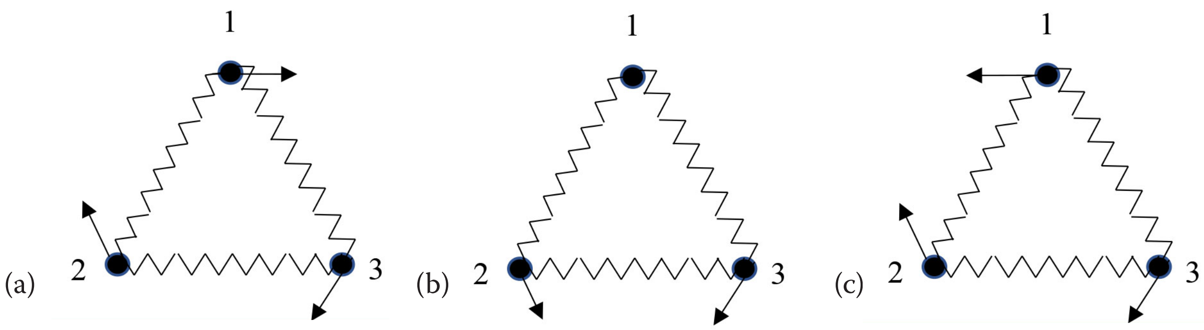


Fig. 2. Displacements corresponding to normal modes of oscillation

The displacements in Fig. 2a correspond to a full-symmetric oscillation (representation $A_1$), and the displacements in Figs. 2b–2c are oscillations with one natural frequency (representation $E$). In this case, the vibrations of $u_s$ and $u_a$ are symmetrical and asymmetrical with respect to the reflection in the plane, which is projected onto a vertical line in the figure. Normal coordinates can be expressed in terms of displacements $x_1$, $x_2$ and $x_3$:

$$u_0 = \frac{1}{\sqrt{3}}\left(x_1 + x_2 + x_3\right),$$
$$u_s = \frac{1}{\sqrt{2}}\left(x_2 - x_3\right),$$
$$u_a = \frac{1}{\sqrt{6}}\left(2x_1 - x_2 - x_3\right).$$

Note that for $k = 0$, the considered model consists of three unconnected oscillators. As it is easy to show, in this case all three frequencies of free normal oscillations coincide.

In the new variables, the system of equations (8) is reduced to the form:

$$\begin{aligned}
&\dot{u}_0 = w_0,\\
&\dot{u}_s = w_s,\\
&\dot{u}_a = w_a,\\
&\dot{w}_0 = -\omega_0^2 u_0 - \gamma w_0 +\\
&+\frac{f}{\sqrt{3}}\left(\cos\varphi\left(\cos\varphi_1 + \cos\varphi_2 + \cos\varphi_3\right) - \sin\varphi\left(\sin\varphi_1 + \sin\varphi_2 + \sin\varphi_3\right)\right),\\
&\dot{w}_s = -\omega_0^2 u_s - \gamma w_s - 3ku_s +\\
&+\frac{f}{\sqrt{2}}\left(\cos\varphi\left(\cos\varphi_2 - \cos\varphi_3\right) - \sin\varphi\left(\sin\varphi_2 - \sin\varphi_3\right)\right),\\
&\dot{v}_a = -\omega_0^2 u_a - \gamma w_a - 3ku_a +\\
&+\frac{f}{\sqrt{6}}\left(\cos\varphi\left(2\cos\varphi_1 - \cos\varphi_2 - \cos\varphi_3\right) - \sin\varphi\left(2\sin\varphi_1 - \sin\varphi_2 - \sin\varphi_3\right)\right),\\
&\dot{\varphi} = \omega.
\end{aligned} \tag{9}$$

As can be seen from the resulting system of equations, the amplitude of steady-state oscillations of different symmetry depends on the phase $\varphi_1$, $\varphi_2$ and $\varphi_3$. In particular, at $\varphi_1 = \varphi_2 = \varphi_3$, only full-symmetric oscillations are excited. On the contrary, with ratios $\varphi_2 = -\varphi_3 = \varphi_1 \pm 2\pi/3$, only oscillations with coordinates $u_s$ and $u_a$ are excited.

The normal oscillations of $u_s$ and $u_a$ can be considered as oscillations of a two-dimensional harmonic oscillator. In quantum theory, the excited state of such an oscillator is completely determined by two constants: the energy of the state $E$ and the angular momentum $M$ (Messiah 1999). In general, several states with different values of $M$ can correspond to one energy level. Free oscillations with coordinates $u_s$ and $u_a$ have the same natural frequency; however, the phases of these oscillations may not coincide, which is analogous to the degeneracy of a quantum two-dimensional oscillator. In this case, an additional parameter that determines the state of the excited system of oscillators is the angular momentum, which in this case can be determined as follows:

$$M = u_s w_a - u_a w_s.$$

Since the choice of coordinates $u_s$ and $u_a$ depends on the choice of the plane of symmetry, it is convenient to convert the angular momentum to the original coordinates and velocities, resulting in the expression:

$$M = \frac{1}{6}\left((v_3 - v_2)x_1 + (v_1 - v_3)x_2 + (v_2 - v_1)x_3\right). \tag{10}$$

The magnitude of the angular momentum depends both on the parameters that determine the free oscillations of the system ( $\omega_0, k, \gamma$ ) and on the parameters of external forces ( $f, \omega, \varphi_1, \varphi_2, \varphi_3$ ). When studying the dependence of the angular momentum on the phases of external forces, one of the values can be set to zero without loss of generality, for example, $\varphi_1 = 0$. Then, away from resonance, when $\left|\omega_0^2 - \omega^2 + 3k\right| >> \gamma\omega$ the dependence on the phases and amplitude of external forces takes a simple form:

$$M = Cf^2\left(\sin\varphi_2 - \sin\varphi_3 - \sin(\varphi_2 - \varphi_3)\right), \quad (11)$$

where the constant $C$ depends on the parameters $\omega_0, \omega, \gamma$ .

## A system of Duffing oscillators in the absence of coupling

Numerical solutions of equations (8) were obtained for a specific type of oscillator forces depending on the displacements:

$$F(x) = x - x^3 . \quad (12)$$

Numerical calculation shows that the system of equations (8), as for a single Duffing oscillator, can have chaotic solutions in a certain range of the parameter $k$. These solutions can be described in terms of probability using the probability density $\rho(x_1, x_2, x_3, v_1, v_2, v_3, \varphi)$, which is determined by an equation similar to equation (6):

$$\omega\frac{\partial\rho}{\partial\varphi} + v_1\frac{\partial\rho}{\partial x_1} + (x_1 - x_1^2 - k\left(2x_1 - x_2 - x_3\right) - \gamma v_1 + f\cos(\varphi + \varphi_1))\frac{\partial\rho}{\partial v_1} +$$
$$+v_2\frac{\partial\rho}{\partial x_2} + (x_2 - x_2^2 - k\left(2x_2 - x_1 - x_3\right) - \gamma v_2 + f\cos(\varphi + \varphi_2))\frac{\partial\rho}{\partial v_2} +$$
$$+v_3\frac{\partial\rho}{\partial x_3} + (x_3 - x_3^2 - k\left(2x_3 - x_2 - x_1\right) - \gamma v_3 + f\cos(\varphi + \varphi_3))\frac{\partial\rho}{\partial v_3} = 0.$$

Let us first consider the case when the oscillators are not connected by elastic forces ($k = 0$). In this case, probability density can be expressed in terms of probability densities for each of the oscillators:

$$\rho(x_1, x_2, x_3, v_1, v_2, v_3, \varphi) = \frac{1}{2\pi}\rho_0(x_1, v_1, \varphi + \varphi_1)\rho_0(x_2, v_2, \varphi + \varphi_2)\rho_0(x_3, v_3, \varphi + \varphi_3), \quad (13)$$

where $\rho_0(x, v, \varphi)$ is the solution of equation (6) for the function $F(x)$ defined by expression (12).

When calculating the average values using the expression (12), it should be taken into account that probability densities are normalized by one for any value $\varphi$:

$$\int dxdv\rho(x, v, \varphi) = 1.$$

As a result of these conditions, the average value of a function that depends only on the variables of one oscillator $\left\langle A(x_i, v_i)\right\rangle$ does not depend on the parameters $\varphi_1$, $\varphi_2$, $\varphi_3$. It follows that the quasienergy defined by expression (7) for oscillators not coupled by elastic forces does not depend on phases $\varphi_1$, $\varphi_2$, $\varphi_3$ either.

On the contrary, the average values of the quantities, which are products of variables related to different oscillators, turn out to depend on the parameters $\varphi_1$, $\varphi_2$, $\varphi_3$. In particular, for the average value of the angular momentum, we obtain:

$$\bar{M}(\varphi_1, \varphi_2, \varphi_3) = \frac{1}{6}\int\frac{d\varphi}{2\pi}(\bar{x}(\varphi + \varphi_1)(\bar{v}(\varphi + \varphi_3) - \bar{v}(\varphi + \varphi_2)) +$$
$$+\bar{x}(\varphi + \varphi_2)(\bar{v}(\varphi + \varphi_1) - \bar{v}(\varphi + \varphi_3)) + \bar{x}(\varphi + \varphi_3)(\bar{v}(\varphi + \varphi_2) - \bar{v}(\varphi + \varphi_1))). \quad (14)$$

Here, the average values of coordinates and velocities are calculated using the probability density $\rho_0(x, v, \varphi)$:

$$\bar{x}(\varphi+\varphi_i)=\int dxdv\rho_0(x,v,\varphi+\varphi_i)x, \quad \bar{v}(\varphi+\varphi_i)=\int dxdv\rho_0(x,v,\varphi+\varphi_i)v .$$

Thus, two different methods can be used to calculate the average value of the angular momentum in this case. In the first method, calculations are performed directly by solving the system of equations (8) over a sufficiently large time interval, followed by time averaging. In the second method, by solving a system of equations for one oscillator over a sufficiently large time interval, the probability density $\rho_0(x,v,\varphi)$ is determined and then the formula (14) is used.

In a numerical experiment, the dependence of the average value of the angular momentum was studied at $\varphi_1=0, \quad \varphi_2=\frac{2\pi}{3}s, \quad \varphi_3=-\frac{2\pi}{3}s$, at $s\in[-1,1]$. The corresponding dependence graphs $\bar{M}(\sigma)$, where $\sigma=\text{sign}(s)\sqrt{|s|}$, are shown in Fig. 3.

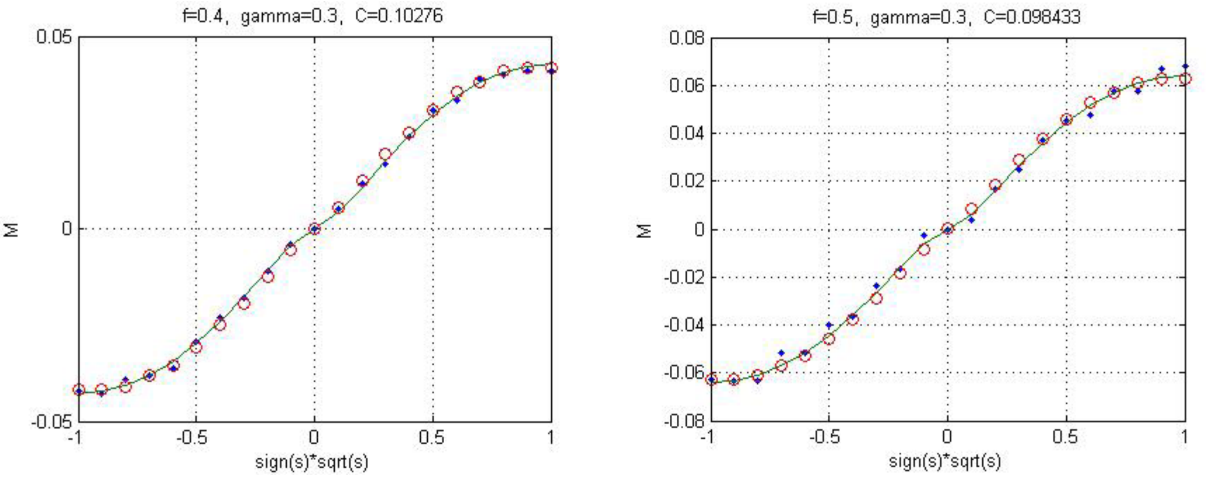


Fig. 3. The average values of the angular momentum when changing the parameters $\varphi_1, \varphi_2, \varphi_3$

Dots on the graphs indicate the values obtained by the first method (averaging over time when solving equations for three oscillators), and circles indicate the values obtained by the second method (averaging with the density matrix of one oscillator according to formula (14)). To compare with the results of the dependence for the case of harmonic oscillators, formula (11) was used, where the constant *C* was determined by the least squares method. The corresponding curve is represented by a solid line. It should be noted that the values of the constants *C* for different values of the parameters *f* turn out to be close.

The results show that the numerical calculations made by various methods coincide quite well, and the dependence on phases $\varphi_1, \varphi_2, \varphi_3$ and amplitude *f* is similar to the dependence obtained by analytical methods for the case of a harmonic oscillator.

In this case, states with a different set of phases correspond to the same value of quasienergy; we can therefore point out degenerate states, similar to what takes place in systems described by quantum theory. However, comparing the results with the quantum mechanical description, we should note that chaotic states are equivalent to mixed states described using the density matrix in quantum theory. Unlike quantum theory, where the density matrix can be constructed as a bilinear function of stationary states (see, for example, (Landau, Lifshitz 1977)), in the case of chaotic states of nonlinear classical dynamics, the principle of superposition of states is inapplicable. Therefore, it is impossible to represent the probability density as a superposition of functions that transform according to some irreducible representation of the symmetry group. Nevertheless, it can be argued that at a value $s=0$, the density matrix $\rho(x_1,x_2,x_3,v_1,v_2,v_3,\varphi)=\frac{1}{2\pi}\rho_0(x_1,v_1,\varphi)\rho_0(x_2,v_2,\varphi+\frac{2\pi s}{3})\rho_0(x_3,v_3,\varphi-\frac{2\pi s}{3})$ corresponds to a fully symmetric state in which the average value of the angular momentum is zero.

As the modulus of the value *s* increases, the modulus of the value of the average angular momentum increases, which means that a component appears in the mixture of states that transforms according to the representation *E* of the symmetry group $C_{3v}$. In the harmonic approximation, as follows from equations (8), a full-symmetric oscillation with zero angular momentum is not excited at the phase ratio

$\varphi_2 = -\varphi_3 = \varphi_1 \pm 2\pi/3$. In a chaotic regime, it can only be argued that the proportion of a full-symmetric oscillation in the probability density takes on a minimum value at the phase ratio $\varphi_2 = -\varphi_3 = \varphi_1 \pm 2\pi/3$.

## Splitting of the average values of quasienergy at $k \neq 0$

As follows from equations (8), the three natural oscillation frequencies of the system for a harmonic oscillator coincide. When the value $k \neq 0$, the frequency of the full-symmetric oscillation becomes different from the frequency of the symmetry oscillation *E*, thus having a degeneracy equal to two. The difference in frequency values increases with the growth of the parameter *k*. In quantum theory, the corresponding phenomenon is called splitting of the energy level (or quasienergy level under external periodic influence). In this case, with a chaotic oscillation, there can be no discreteness of quasienergy values, of course. However, it is possible to investigate the change in the average value of quasienergy with an increase in the value of *k*, starting from zero. The results of the corresponding numerical experiment are shown in Fig. 4.

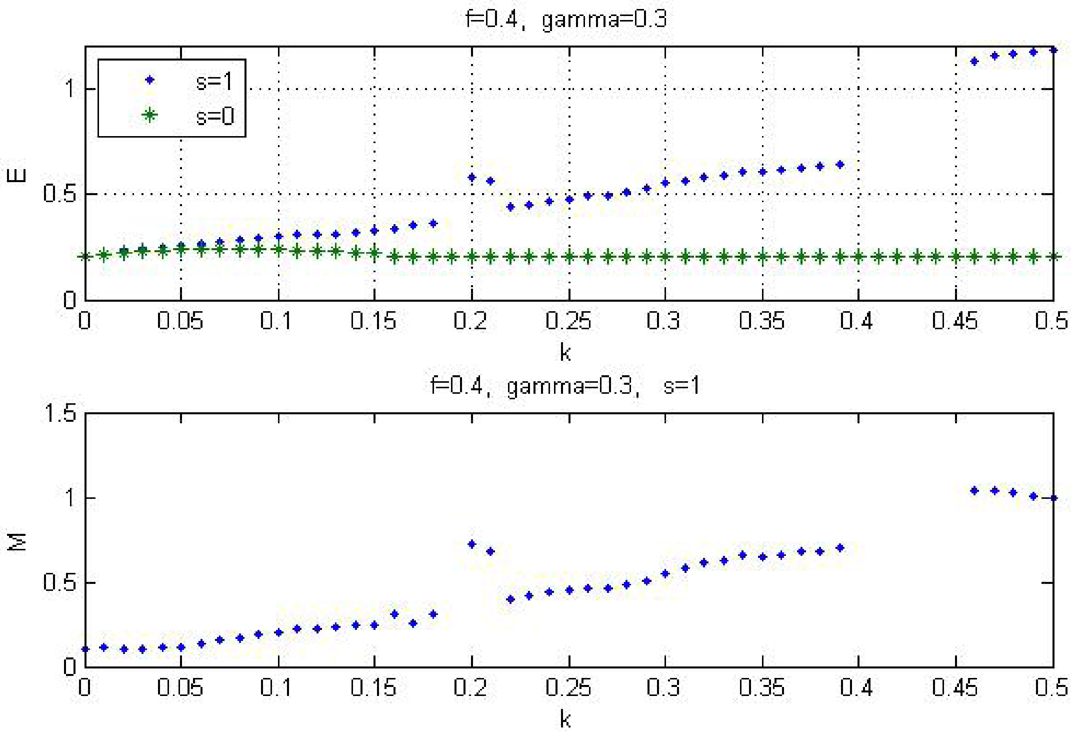


Fig. 4. Quasienergy and angular momentum depending on the magnitude of the interaction of the oscillators *k*

The upper figure shows the quasienergy values at the phase ratio $\varphi_1 = \varphi_2 = \varphi_3$ ($s = 0$), which corresponds to the zero value of the angular momentum, and the quasienergy values at the phase ratio $\varphi_2 = -\varphi_3 = \varphi_1 \pm 2\pi/3$ ($s = 1$), which corresponds to the maximum value of the average angular momentum. The lower figure shows the values of the average angular momentum at $s = 1$ and the corresponding values of the parameter *k*. Discontinuities in the graphs arise due to the fact that in the regions of the corresponding values of the parameter *k*, the solutions of the equations are not chaotic but regular (periodic).

## Conclusion

It may seem strange that the results obtained for regular solutions in the case of interaction of an external field with a system of harmonic oscillators turn out to be similar in the case of nonlinear Duffing oscillators with chaotic solutions. In fact, this is largely due to the properties of the symmetry of the system. Similar conclusions can be obtained, for example, by examining a quantum mechanical system of three nonlinear oscillators located in an external field similar to the one discussed above. The density matrix of such a system, which determines the population of an excited degenerate level, turns out to be proportional to the square of the matrix element $\langle 0|V|i\rangle$, where *V* is the operator of interaction with an external field, 0 corresponds to a non-degenerate ground state, and *i* corresponds to one of the degenerate excited states due to symmetry. The corresponding equations for the density matrix are derived, for example, in the monograph (Blum 2012). For the case considered here, the operator of interaction with an external field can be written as:

$$V(t) = \sum_j d_j f \cos(\omega t + \varphi_j),$$

where $d_j$ is the operator of the dipole moment of the $j$-th oscillator. Further calculation of the density matrix using symmetry properties leads to an average value of the angular momentum, with a dependence similar to (11).

## Conflict of Interest

The authors declare that there is no conflict of interest, either existing or potential.

## Author Contributions

All the authors discussed the final work and took part in writing the article.

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