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Black hole shadow: Experimental test of different models and shadow of dynamical Hayward black hole

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Abstract. Recent observations of black hole shadows have revolutionized our ability to probe gravity in extreme environments. This manuscript presents a novel analytic model of the shadow of a dynamical Hayward black hole. We show that under some choice of mass and regularization functions, this spacetime admits a homothetic Killing vector, which allows the reduction of the second-order differential equations of motion to the first-order ones. After that, we introduce the coordinate transformation to conformally-static coordinates and introduce a new conserved quantity along null geodesics. We prove that in the dynamical case the regularization parameter always decreases the radius of a photon sphere. Inspired by the experimental data obtained by the Event Horizon Telescope Collaboration, we compare the obtained images of Sagittarius A^* with a black hole shadow in Reissner–Nordstrom, Bardeen and Hayward black holes, respectively.

Keywords: black hole, photon sphere, mass-dependent, shadow, dynamical Hayward black hole

Introduction

In 1915, Albert Einstein presented general relativity. One year later, Schwarzschild solved his equations for empty spherically-symmetric spacetime, and his solution became known as Schwarzschild metric. Later, it was understood that Schwarzschild spacetime describes a static spherically-symmetric black hole. However, a black hole distorts spacetime so strongly that even light cannot escape it. That is why one can learn about properties of a black hole through its influence on surrounding matter. For this reason, it becomes really important to study the properties of motion around these objects. Thorough investigation of light propagation revealed that light can move along an unstable circular orbit which forms a black spot on the observer's sky, which is known as a black hole shadow (Bardeen et al. 1972; Synge 1966;).

In 2019, the Event Horizon Telescope Collaboration published images of the supermassive black hole located in the center of galaxy $M87$ (Alberdi et al. 2019). Later, in 2022, they published images of a supermassive black hole, Sagittarius A^*, in our own galaxy — Milky Way (Akiyama et al. 2022). These observation data endowed a black hole with the status of a real astrophysical object and drew huge attention of the scientific community.

The idea that this shadow could be observed was brought forward in 2000 in a pioneering paper by Falcke et al. (Falcke et al. 2000). Tsupko et al. showed that a shadow can be used as a cosmological ruler (Tsupko et al. 2020). The problem of using the shadow as a cosmological ruler is discussed in the paper (Vagnozzi et al. 2020). The black hole shadow can be used to distinguish different black hole models and even to find differences between general relativity and alternative theories of gravity (Vagnozzi et al. 2023). The review of an analytical study of static black hole shadows is given in (Perlick, Tsupko 2022). The influence of plasma on the observed size of a black hole is considered in the paper (Perlick et al. 2015). New analytical methods of studying black hole shadows have been developed in papers (Vertogradov, Övgün 2024a; 2024b; Vertogradov et al. 2024). However, all these models describe the static black hole; the astrophysical black hole, however, changes its mass while absorbing and emitting matter. Thus, the real black hole should be described by dynamical spacetime. There is a numerical method of calculating a dynamical black hole shadow developed in (Koga et al. 2022; Mishra et al. 2019), but analytical models are scarce (Heydarzade, Vertogradov 2024; Solanki, Perlick 2022; Tsupko, Bisnovatyi-Kogan 2020). The problem with an analytical model is that there is only one conserved quantity in the spherically symmetric black hole spacetime. It is an angular momentum-per-mass L. So, one needs to seek extra symmetry to reduce the second-order differential equations of motion to the first one. Dynamical spacetime does not possess the timelike Killing vector $\frac{\partial}{\partial t}$, so one should look for conformal Killing vectors. If the spacetime admits the conformal Killing vector, then there is an extra conserved quantity along null geodesics. The shadow formation in Vaidya and charged Vaidya spacetime has been investigated in the papers (Heydarzade, Vertogradov 2024; Solanki, Perlick 2022). Vaidya solution (Vaidya 1951) is one of the exact dynamical solutions of the Einstein equations. It can be regarded as a dynamical generalization of the static Schwarzschild solution. The Vaidya spacetime is widely used in many astrophysical applications with strong gravitational fields. In general relativity, this spacetime assumed greater importance with the completion of the junction conditions at the surface of the star by Santos (Santos 1985). The horizon structure and surface gravity of this solution have been investigated in (Nielsen 2014, Nielsen, Yoon 2008). The Vaidya spacetime can be extended to include both null dust and null string fluids leading to the generalized Vaidya spacetime (Wang, Wu 1999). A detailed investigation of the properties of the generalized Vaidya spacetime can be found in (Glass, Krisch 1998; Glass, Krisch 1999; Husain 1996; Vertogradov 2024). Charged Vaidya spacetime (Bonnor, Vaidya 1970) has been widely investigated in gravitational collapse and naked singularity formation (Beesham, Ghosh 2003; Lake, Zannias 1991; Patil et al. 1999; Vertogradov 2022;). The conformal symmetry of the charged Vaidya spacetime and Hawking radiation have been considered in (Ibohal, Kapil 2010; Koh et al. 2024; Ojako et al. 2020; Vertogradov, Kudryavcev 2023). Surrounded Vaidya spacetimes with cosmological fields have been obtained in papers (Heydarzade, Darabi 2018a; Heydarzade, Darabi 2018b; Heydarzade, Darabi 2018c). The process of extracting energy from the charged Vaidya black hole has been considered in (Vertogradov 2023).

In this paper, we compare experimental images obtained by the Event Horizon Telescope Collaboration from Sagittarius A^* with well-known black hole models, i. e. the electrically charged black hole described by Reissner — Nordstrom spacetime and two regular black holes described by Bardeen and Hayward metrics. Then we show that there are several values of extra parameters under which a shadow corresponds to experimental data. We also present the shadow model of a dynamical Hayward regular black hole. We consider the linear mass and regularization functions because Hayward spacetime with these choices of functions admits a homothetic Killing vector. We introduce the coordinate transformation to the conformally static coordinates in which we introduce an extra conserved quantity and estimate the influence of the regularization parameter on the visible size of a shadow.

This paper is organized as follows: in Section 2 we briefly describe the method of black hole shadow calculation and numerically calculate the shadow for Reissner — Nordstrom, Bardeen and Hayward spacetimes. Then we compare the obtained shadow with the image of Sagittarius A^*. In Section 3 we analytically calculate the radius of the photon sphere in dynamical Hayward spacetime and estimate the influence of the regularization parameter on it. Section 4 discusses the obtained results.

The geometrized system of units $c = 1 = G$ will be used throughout the paper. We also use the signature $-+++$.

Shadow of spherically-symmetric black holes

Astrophysical black holes are supposed to be rotational. The shadow of these objects depends on the angle of observation. However, if an observer sees the shadow along the rotational axis, then the shadow has the same form and properties as a static spherically-symmetric one. For this reason, we compare images obtained by the Event Horizon Telescope Collaboration with several well-known solutions of the Einstein field equations describing non-rotational black holes. Without loss of generality, the spherically symmetric spacetime can be written as

$$ds^2 = -f dt^2 + f^{-1} dr^2 + r^2 d\Omega^2, \tag{1}$$

where $f \equiv f(r)$ is the lapse function and $d\Omega^2 = d\theta^2 + \sin^2\theta d\varphi^2$ is the metric on the unit two-sphere. The spacetime (1) admits two constants of motion, i. e. energy-per-mass E and angular momentum-per-mass L which are given by the equatorial plane $\theta = \frac{\pi}{2}$

$$E = f\frac{dt}{d\lambda},$$
$$L = r^2\frac{d\varphi}{d\lambda}, \tag{2}$$

where λ is an affine parameter. By using the null geodesic condition $g_{ik}u^i u^k = 0$, one can write the radial component of the geodesic motion equation as

$$\left(\frac{dr}{d\lambda}\right)^2 + V_{eff} = 0,$$
$$V_{eff} = L^2\frac{f}{r^2} - E^2. \tag{3}$$

Here V_{eff} has effective potential. A black hole shadow is formed by a photon sphere, which is composed of unstable circle orbits. For a circle orbit, there are two conditions $\frac{dr}{d\lambda} = \frac{d^2r}{d\lambda^2} = 0$ which lead to

$$V_{eff}(r_{ph}) = 0,$$
$$V'_{eff}(r_{ph}) = 0, \tag{4}$$

where r_{ph} is the radius of a photon sphere and prime denotes the radial derivative. For Schwarzschild spacetime, i. e.

$$f(r) = 1 - \frac{2M}{r}, \tag{5}$$

the radius of a photon sphere $r_{ph} = 3M$. The radius of a shadow is defined from the first condition (4)

$$b \equiv \frac{L}{E} = \frac{r_{ph}}{\sqrt{f(r_{ph})}}, \tag{6}$$

where b is an impact parameter. For a Schwarzschild black hole, it has the value of $b = 3\sqrt{3}M$. The angular size ω_{sh} of a shadow is given by

$$\sin^2\omega_{sh} \approx \frac{b^2}{r_o}, \tag{7}$$

where r_0 is the distance between an observer and a black hole. The angular size ω_{sh} of a Schwarzschild black hole for a Sagittarius a^* is ~53μ as. However, the observational data shows that the real angular size of Sagittarius a^* is ~52μ as. This indicates that one should consider black hole models different from Schwarzschild ones. It is a well-known fact that the electrical charge of a black hole decreases its angular size. The metric that describes an electrically charged black hole is Reissner–Nordstrom spacetime

$$f(r) = 1 - \frac{2M}{r} + \frac{Q^2}{r^2}, \tag{8}$$

where Q is an electrical charge of a black hole with dimension of length. The radius of a photon sphere in this case is

$$r_{ph} = \frac{1}{2}\left(3M + \sqrt{9M^2 - 8Q^2}\right). \tag{9}$$

As one can see that if $Q \to 0$, then we obtain the radius of a photon sphere like in Schwarzschild spacetime $r_{ph} = 3M$. However, if one considers other spacetimes, like regular Bardeen (Bardeen 1968) and Hayward (Hayward 2006) ones, which are given by

$$f(r) = 1 - \frac{2Mr^2}{(r^2 + g^2)^{\frac{3}{2}}} \tag{10}$$

and

$$f(r) = 1 - \frac{2Mr^2}{r^3 + 2ML^2} \tag{11}$$

respectively, then one will see that it is impossible to find exact analytical expression for the radius of a photon sphere and angular size. However, by using the method developed in (Vertogradov, Övgün 2024a), one can estimate the influence of parameters g and L on the angular size of a black hole. To use this method, we write the lapse function in the form

$$f = \left(1 - \frac{2M}{r}\right) e^{\xi(r)}, \tag{12}$$

where ξ is a small arbitrary function. If $\xi'(3M) < 0$, then minimal geometrical deformation decreases the radius of a photon sphere, and if $\xi(3M) > 0$, then the angular size of a black hole shadow decreases. In the paper (Vertogradov, Övgün 2024b) it was shown that parameters g in Bardeen spacetime (10) and L in Hayward spacetime (11) decrease the angular size of a black hole shadow. It means we should find such values Q, g and L that the shadow size is ~52μ as.

Now let us look at the effectiveness of the presented models. Let us calculate the angular size of the shadow of the black holes M87* and Sgr A*.

Schwarzschild black hole

As it is known, the radius of the shadow of a black hole is

$$V_{eff}(r_{ph}) = 0 \to \frac{\mathcal{L}^2}{r_{ph}^2} A(r_{ph}) - E^2 = 0 \to \frac{b^2}{r_{ph}^2} A(r_{ph}) - 1 = 0. \tag{13}$$

Let us consider the shadow of a spherically symmetric Schwarzschild black hole. Then the metric has the form:

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \left(1 - \frac{2M}{r}\right) dr^2 + r^2 d\Omega^2. \tag{14}$$

Substituting the value into the effective potential, we get

$$V_{eff} = \left(1 - \frac{2M}{r}\right) \frac{L^2}{r^2} - E^2. \tag{15}$$

When substituting the condition, we get:

$$\frac{dV_{eff}}{dr} = 0 \to A' - 2A = \frac{2M}{r} - 2 + \frac{4M}{r} = 0 \to r_{ph} = 3M. \tag{16}$$

Thus, substituting the radius into the condition and resolving with respect to the impact parameter, we get:

$$V_{eff} = 0 \to \left(1 - \frac{2M}{r}\right)\frac{L^2}{r^2} - E^2 = 0 \to$$
$$\left(1 - \frac{2M}{r}\right)\frac{b^2}{r^2} - 1 = 0 \to b_{cr} = 3\sqrt{3}M. \tag{17}$$

Experimental data have shown that the mass of SqrA* is equal to $4.3 \times 10^6 M_o$, and M87* — $3 \times 10^9 M_o$. Mass and charge are included in a geometrized system of units, that is, in a system in which G and c are equal to 1. Accordingly, they have a dimension in centimeters. To convert kilograms and pendants into centimeters, we need to use the following formula:

$$m = \frac{GM}{c^2}. \tag{18}$$

Substituting the values, we find that the mass of the black hole SgrA* corresponds to

$$M = 6 \times 10^{11} g, \tag{19}$$

and M87* —

$$M = 4 \times 10^{14} g. \tag{20}$$

Now let us convert r_0 to centimeters; then for SgrA* it will be equal to

$$r_0 = 26 \times 10^{21} cm, \tag{21}$$

and for M87* —

$$r_0 = 56 \times 10^{24} cm. \tag{22}$$

The Schwarzschild solution is not successful, since the values are somewhat overestimated, and it is impossible to use an additional parameter to reduce the diameter. Therefore, the Schwarzschild metric is good for research as an ideal model for qualitative effects, but it is not suitable for description, and other models need to be considered.

Reissner — Nordstrom black hole

The metric is spherically symmetric and static.

In the case of Reissner–Nordstrom, this is a known quantity that can be found when solving the second equation with respect to r_{ph}. We get the following expression:

$$r_{ph} = \frac{3}{2}\left(M + \sqrt{M^2 - \frac{8}{9}Q^2}\right). \tag{23}$$

The first condition implies that

$$V_{eff}(r_{ph}) = 0. \tag{24}$$

Then we express the impact parameter b:

$$b_{cr} = \sqrt{\frac{(3M + \sqrt{9M^2 - 8Q^2})^4}{24M^2 + 8M\sqrt{9M^2 - 8Q^2} + 48Q^2}}. \tag{25}$$

To correctly estimate the angular diameter, we use the expression

$$2\alpha = \frac{b}{r_0}, \tag{26}$$

where r_0 — distance to SqrA* and M87*.

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Now let us substitute the masses and r_0 of M87* and SgrA* in the expression for the angular parameters. We assume that the q, the gravitational charge, is 0.01M, 0.1M, 0.5M, 0.75M, 0.99M and get the corresponding angular sizes.

The highest values are reached at 0.01M. To understand what 0.01M is for a charge and what is a charge in centimeters, we convert the charge into coulombs according to the following formula:

$$Q^2 = \frac{q^2 G}{4\pi\varepsilon_0 c^4}. \tag{27}$$

To do this, we will use the SI-system, in which the mass is in meters and the charge is in coulombs. Thus, the value has the form

$$Q = 7 \times 10^{25}. \tag{28}$$

The charge is gigantic and corresponds to a supermassive black hole of the order of millions of solar masses.

Hayward black hole

The metric describes a regular black hole. Similary to the Reissner–Nordstrom case, we obtain an expression for the radius of the photon sphere:

$$r^6 - 3Mr^5 + 4L^2Mr^3 + 4L^4M^2 = 0. \tag{29}$$

Numerically solving it and substituting the value, we express the impact parameter b:

$$b_{cr} = \sqrt{\frac{r^5 + 2r^2ML^2}{r^3 + 2ML^2 - 2Mr^2}}. \tag{30}$$

Consider the upper bound of the range of possible values of the regularization parameter L, at which an extreme black hole occurs. The mass of the black hole must be less than or equal to zero.

As we know, the metric describing Hayward is presented as follows:

$$f(r) = 1 - \frac{2Mr^2}{r^3 + 2ML^2}. \tag{31}$$

Let us bring it to a common denominator and take the derivative in r. As a result, we take a value greater than zero:

$$r = \frac{4M}{3}. \tag{32}$$

Substituting r into the expression for the metric and solving with respect to L, we get the boundary for an extreme black hole:

$$L \leq \frac{4M}{3\sqrt{3}}. \tag{33}$$

Since at a higher value there is no horizon, or the black hole itself, and the result in the form of a shadow of a black hole is also lost, we are interested in values ranging from 0 to $\frac{4m}{3\sqrt{3}}$. The impact parameter L allows us to get the most correct result, which can be seen with L equal to 0.01M.

Bardeen black hole

The Bardeen metric also describes regular black holes.

By analogy with the previous metrics, we get the expression for r_{ph}:

$$r^{10} + (5g^2 - 9M^2)r^8 + 10g^4r^6 + 10g^6r^4 + 5g^8r^2 + g^{10} = 0. \tag{34}$$

Based on its numerical solution, we obtain the impact parameter b:

$$b_{cr} = \sqrt{\frac{r^2(r^2+g^2)^{\frac{3}{2}}}{(r^2+g^2)^{\frac{3}{2}} - 2Mr^2}}, \tag{35}$$

g — the magnetic charge of the monopole characterizing the current model. Let us find its upper bound.

Bardeen's metric:

$$f(r) = 1 - \frac{2Mr^2}{(r^2+g^2)^{\frac{3}{2}}}. \tag{36}$$

We bring it to a common denominator and take the derivative of r in the same way as Hayward. We get two roots, and by solving one of them we come to the following value:

$$r = \sqrt{\frac{16M^2 - 9g^2}{9}}. \tag{37}$$

Substituting the result into the Bardeen metric, we find g:

$$g \leq \frac{4M}{3\sqrt{6}}. \tag{38}$$

Iterating through the values in the range from 0 to $\frac{4m}{3\sqrt{6}}$, we get the results.

Since the charge is small, as for Reissner–Nordstrom, they differ little from each other. Nonlinear electrodynamics was introduced to describe powerful electric fields.

Table 1. Experimental data

Schwarzschild	–	–	–
M87*	18	–	–
SgrA*	53	–	–
Reissner–Nordstrom	Q = 0.01m	Q = 0.1m	Q = 0.99m
M87*	17	16	6
SgrA*	53	52	19
Hayward	L = 0.01m	L = 0.1m	$L_{max} = \frac{4m}{3\sqrt{3}}$
M87*	17	17	16
SgrA*	52	52	48
Bardeen	G = 0.01m	G = 0.1m	$G_{max} = \frac{4m}{3\sqrt{6}}$
M87*	17	17	16
SgrA*	52	52	44

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Shadow of a dynamical Hayward black hole

Properties of the Hayward metric

In the general case, the metric for a regular black hole is written as:

$$ds^2 = -F(r)dt^2 + \frac{dr^2}{F(r)} + r^2 d\Omega^2. \tag{39}$$

In the article (Hayward 2006) Hayward introduced $F(r)$ as $F(r) = (1 - \frac{2Mr^2}{r^3 + 2ML^2})$. Here, if the regularization parameter L tends to zero, Hayward spacetime becomes a Schwarzschild one. This L prevents scalar invariants from divergence at $r \to 0$. Hayward spacetime at $r \to 0$ has de Sitter core.

$$F(r) \sim 1 - \frac{M}{r}, r \to \infty$$
$$F(r) \sim 1 - \frac{r^2}{L^2}, r \to 0\ . \tag{40}$$

A black hole without singularity in the center is called a regular black hole, and the Hayward black hole is one of the examples of models of such black holes. Let us write down Hayward metric and define some characteristics:

$$ds^2 = -\left(1 - \frac{2Mr^2}{r^3 + 2ML^2}\right)dt^2 + \frac{dr^2}{\left(1 - \frac{2Mr^2}{r^3 + 2ML^2}\right)} + r^2 d\Omega^2. \tag{41}$$

First of all, we should show that the metric is really regular. For this purpose, we calculate Kretschmann scalar:

$$K = \frac{24}{L^4}\left(1 - \frac{r^3}{ML^2}\right). \tag{42}$$

From this, we can see that $r \to 0$, which corresponds to the center of a black hole, does not lead to a divergence of scalar invariants and the emergence of a singularity. Another important property of the metric is that a black hole has two horizons — the inner and the outer one.

Under some values of M and L this black hole becomes an extremal one, i. e. these two horizons merge. This means that the Hayward metric can describe an extreme black hole. Formation of an extreme black hole is possible under the following conditions: $M = \frac{3\sqrt{3}}{4}L$.

The shadow of a Hayward black hole

In order to describe a black hole shadow, one needs to find a photon sphere. For this purpose we need to find a solution at which photons move in a circular orbit in the black hole potential. The Lagrangian for the metric (41) has the following form:

$$\mathfrak{L} = \frac{1}{2}\left[-\left(1 - \frac{2Mr^2}{r^3 + 2ML^2}\right)\frac{dt^2}{d\lambda^2} + \left(1 - \frac{2Mr^2}{r^3 + 2ML^2}\right)\frac{dr^2}{d\lambda^2} + r^2\frac{d\Omega 2}{d\lambda^2}\right]. \tag{43}$$

Using this Lagrangian, we can find the effective potential V_{eff}:

$$V_{eff} = E^2 - \left(1 - \frac{2Mr^2}{r^3 + 2ML^2}\right)\frac{\mathcal{L}}{r^2}. \tag{44}$$

From (44) and the condition for the existence of circular orbits, we have an expression for the shadow of the Hayward black hole:

$$r^6 - 3Mr^5 + 4L^2Mr^3 + 4L^4M^2 = 0. \tag{45}$$

It is easy to see that if we put $L = 0$, the radius of a photon sphere becomes the Schwarzschild one $r_{ph} \to 3M$.

Dynamical shadow of the Hayward black hole

In the previous subsection, we described the static model. In the static case, the effects associated with accretion and radiation are not taken into account, which is well suited for a superficial study of the model and properties of the black hole. However, for a deeper understanding of this object, it is necessary to set a task closer to reality. The obtained experimental data are based on the fact that telescopes detect X-ray emission, which is caused by the friction of the plasma that forms the accretion disk, and the model has to be improved to take into account the effects associated with the accretion of matter.

We use the Eddington–Finkelstein coordinates $\{v, r, \theta, \varphi\}$, in which the Hayward spacetime takes the form:

$$ds^2 = -\left(1 - \frac{2M(v)r^2}{r^3 + 2M(v)L(v)^2}\right)dv^2 + 2dvdr + r^2 d\Omega^2. \tag{46}$$

Since the components of the metric tensor are time-dependent, a time-like Killing vector does not exist, and the solution becomes analytically difficult since the energy is not conserved. To simplify the task, we need to carry out conformal transformations. In the paper (Nielsen 2014) it was proven that the existence of a conformal Killing vector is possible with the following choice of mass and regularization functions:

$$\begin{aligned} M(V) &= \mu v, \\ L^2(v) &= \alpha \xi^2 v^2. \end{aligned} \tag{47}$$

By introducing these transformations into the metric, we obtain:

$$ds^2 = -\left(1 - \frac{2\mu v r^2}{r^3 + 2\mu\alpha\xi^2 v^3}\right) + 2dvdr + r^2 \Omega^2. \tag{48}$$

We also make coordinate transformation:

$$\begin{aligned} v &= r_0 \exp\left(\frac{T}{r_0}\right), \\ r &= Rexp\left(\frac{T}{r_0}\right). \end{aligned} \tag{49}$$

Then considering (48) and (49), we get:

$$\begin{aligned} ds^2 &= \exp\left(\frac{2T}{r_0}\right) ds_1^2, \\ ds_1^2 &= -\left(1 - \frac{2\mu r_0 R^2}{R^3 + 2\mu\alpha\xi^2 r_0^3} - \frac{2R}{r_0}\right)dT^2 + 2dTdR + R^2 d\varphi^2. \end{aligned} \tag{50}$$

Based on the obtained data we can write down the effective potential in the form:

$$V_{eff} = \left(1 - \frac{2\mu r_0 R^2}{R^3 + 2\mu\alpha\xi^2 r_0^3} - \frac{2R}{r_0}\right)\frac{L^2}{R^2} - E^2. \tag{51}$$

This expression has a complex analytical form. For simplicity, we note that we can perform the following substitutions:

$$\frac{2\mu r_0 R^2}{R^3 + 2\mu\alpha\xi^2 r_0^3} = \frac{2\mu r_0}{R} - \varepsilon(R). \tag{52}$$

This leads to the fact that the *F(R)* is reduced to the form:

$$1-\frac{2\mu r_0}{R}-\varepsilon(R)-\frac{2R}{r_0}. \tag{53}$$

And as we know (Heydarzade, Vertogradov 2024), $1-\frac{2\mu r_0}{R}-\frac{2R}{r_0}$ is F(R) for dynamic Vaidya spacetime. And we can describe the effective potential as the sum of two components, F(R) for Vaidya spacetime and $\varepsilon(R)$. Then we need to find $\varepsilon(R)$:

$$\begin{aligned}\varepsilon(R) &= -\frac{2\mu r_0 R^2}{R^3+2\mu\alpha\xi^2 r_0^3}+\frac{2\mu r_0}{R}\\ &= \frac{4\mu^2\alpha\xi^2 r_0^4}{R^4+\mu\alpha\xi^2 r_0^3 R}.\end{aligned} \tag{54}$$

From (53) and (54) we can find the effective potential V_{eff} as:

$$V_{eff}=\left(1-\frac{2\mu r_0}{R}-\frac{2R}{r_0}+\frac{4\mu^2\alpha\xi^2 r_0^4}{R^4+\mu\alpha\xi^2 r_0^3 R}\right)\frac{L^2}{R^2}-E^2. \tag{55}$$

It follows that the radius of the photonsphere can be represented as the sum of the radius of the Vaidya photonsphere and the radius of the function $\varepsilon(R)$:

$$R_{ph}=R_V+\alpha R_\varepsilon. \tag{56}$$

Since the radius of the photon sphere in the Vaidya case is already known, we only need to find R_ε. For this purpose we write down V_{eff}:

$$V_{eff}=b^2 Q(R)\big(1+\alpha Q(R)\big)-1. \tag{57}$$

Functions Q(R) and G(R) are

$$\begin{aligned}Q(R) &= \frac{\left(1-\frac{2\mu r_0}{R}-\frac{2R}{r_0}\right)}{R^2},\\ G(R) &= \frac{4\mu^2\xi^2 r_0^4}{(R^4+\alpha\mu\xi^2 r_0^3 R)}\frac{1}{\left(1-\frac{2\mu r_0}{R}-\frac{2R}{r_0}\right)}.\end{aligned} \tag{58}$$

From (24) we find:

$$Q'(R)\big(1-\alpha G(R)\big)-\alpha Q(R)G'(R). \tag{59}$$

Assuming that α is very small, we write (59) as:

$$Q''(R_0)\alpha R_1-\alpha G'(R_0)Q(R_0)=0. \tag{60}$$

Hence it follows

$$R_\varepsilon=-\frac{G'(R_0)Q(R_0)}{Q''(R0)}. \tag{61}$$

In order to find R_ε, we need to calculate functions $Q''(R)$, $G'(R)Q(R)$. To simplify the calculations, let us assume that $1-\frac{2\mu r_0}{R}-\frac{2R}{r_0}=h(R)$. Then

$$\begin{aligned}Q(R) &= \frac{h(R)}{R^2},\\ G(R) &= \frac{\varepsilon(R)}{h(R)}.\end{aligned} \tag{62}$$

From this expression we can represent the equations in a general form:

$$\begin{aligned} Q'(R) &= \frac{h'(R) - 2h(R)}{R^3}, \\ Q''(R) &= \frac{h''(R)R - h'(R)}{R^3}, \\ G'(R) &= \frac{\varepsilon'(R)h(R) - h'(R)\varepsilon(R)}{h^2(R)}. \end{aligned} \tag{63}$$

Let us use the condition that $h'(R) - 2h(R) = 0$. Then we can make the following transformations:

$$\begin{aligned} G'(R) &= \frac{\varepsilon'(R) - \frac{2}{R}\varepsilon(R)}{h(R)}, \\ Q''(R) &= \frac{h''(R) - 2h(R)}{R^4}, \\ G'Q(R) &= \frac{\varepsilon'(R) - \frac{2}{R}\varepsilon(R)}{R^2}. \end{aligned} \tag{64}$$

Then substituting (61) into (64), we find:

$$R_1 = -\frac{R^2\varepsilon'(R) - 2R\varepsilon(R)}{h''(R)R^2 - h(R)}. \tag{65}$$

And from (64) and (65) we obtain:

$$R_1 = -\frac{12\mu^2\xi^2 r_0^5}{R^4}. \tag{66}$$

Since R_ε is negative, the photonsphere radius R_{ph} will be smaller than R_V. Thus, we can conclude that the regularization parameter reduces the shadow size, which should be taken into account when obtaining experimental data.

Conclusions

The investigation of black hole shadows can give us a better understanding of these mysterious objects. It is a well-established fact that the Schwarzschild metric is good at describing some effects of the black hole, but it does not describe the real astrophysical object. For this reason, one should seek for alternative black hole models and compare them with only accessible experimental data obtained by the Event Horizon Telescope Collaboration. However, even this knowledge allows one to reject several models. Recently, it was proven that a primary hair in hairy Schwarzschild and Reissner — Nordstrom spacetimes obtained by gravitational decoupling (Ovalle et al. 2021) always increases the radius of a photon sphere. It means that we can say that hairy a Schwarzschild black hole does is not suitable for describing the black hole in the center of our galaxy. The primary hair also increases the radius of a photon sphere in a charged version of this solution. However, it can still be a suitable model which demands even more charge than Reissner — Nordstrom, though. In this paper we showed that the Reissner — Nordstrom charged black hole and the regular Bardeen and Hayward black holes might suit as models for describing the supermassive black hole Sagittarius A^* in the center of our galaxy. However, the amount of electrical charge of ~$10^2$6 C sounds unrealistic for a real astrophysical black hole. A real astrophysical black hole is surrounded by an accretion disc. This disc has an influence on the black hole metric, and if we assume constant accretion onto the black hole, then it should change its mass. This fact tells us that the spacetime around these objects should be dynamical. However, we encounter serious problems with constructing a black hole shadow for dynamical spacetimes because in general, it does not have enough constants of motion to reduce the second-order differential equations to the first one although the first steps in this direction have been made in the paper (Vertogradov, Övgün 2024a). Thus, we should seek for extra symmetries to introduce a new conserved quantity along null geodesics. This extra symmetry can be a consequence of the existence of a conformal Killing vector. We have found conformal Killing vector

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for a Hayward regular black hole and introduced a coordinate transformation to conformably-static coordinates. This allowed us to calculate a shadow for dynamical Hayward spacetime. Although a full analytical description is not possible in this spacetime, we can estimate the influence of the extra parameter L on the size of a shadow. Analytically, we were able to prove that this regularization parameter L always decreases the radius of a photon sphere and can serve as a good model for describing the supermassive black hole Sagittarius A^*.

Conflict of Interest

The authors declare that there is no conflict of interest, either existing or potential.

Author Contributions

All the authors discussed the final work and took part in writing the article.

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