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Description of the chaotic state of a nonlinear dynamical system using a smoothed distribution function

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Abstract. This paper considers the possibility of describing the chaotic state of dynamical systems using a distribution function. It is shown that for dissipative systems, a description employing a distribution function similar to that used in statistical physics is inadequate. This is explained by the fact that with long evolution times, the corresponding function ceases to be continuous. A definition for a smoothed distribution function, obtained through a specific averaging of the statistical distribution function, is proposed. The equation for the smoothed distribution function is derived. The results are applied to calculate the emission spectra of dynamical systems in a chaotic state.

Keywords: nonlinear dynamics, chaos, distribution function, probability density, chaotic attractor, fluctuations of the distribution function, radiation spectra

Introduction

A characteristics feature of nonlinear equations is the existence of solutions that demonstrate chaotic behavior. Such equations describe, for example, chaotic vibrations in mechanical problems (Gonchenko et al. 2017; Grinchenko et al. 2007; Kuznetsov 2006; Loskutov 2007; Malinetsky and Potapov 2000; Sagdeev et al. 1988; Schuster 1984). Currently, studies in nonlinear optics, where solutions with chaotic states can also exist, are of significant interest (Gorbacheva and Ryzhov 2022; Ryzhov et al. 2019; 2021a; 2021b; 2024).

In nonlinear dynamics, so-called dissipative systems are of particular interest. These systems are characterized by the fact that the region of phase space in which the system's state evolves contracts over time. Consequently, the system's state tends toward a specific set of points, known as an attractor. Chaotic states in such systems are described by a 'strange' or 'chaotic' attractor. Such attractors have a nontrivial structure that depends on the system's parameters, but not on the initial conditions. A visual representation of a chaotic attractor's structure is provided by graphs of Poincare sections (see, for example, (Kuznetsov 2006)). A characteristic feature of chaotic attractors and their corresponding Poincare sections is their fractional dimension, the specific value of which depends on both the system under consideration and the parameters used in the calculation.

The intricate pattern of Poincare sections for a system's chaotic state in some ways resembles graphs of electron density distribution in atoms and molecules. This suggests that one might attempt to describe the chaotic state of a dynamical system using the language of probability, employing concepts of probability density or a distribution function. This idea was first expressed in the work of Zaslavskii (Sagdeev et al. 1988). An attempt to obtain an equation for the probability density of a specific rotator system in an external periodic field was made in (Liapzev 2019). For this purpose, a limiting transition from the corresponding quantum mechanical problem to a classical description was used. The resulting equation at $t\to\infty$ is a linear partial differential equation. It was shown that for some special cases, solving this equation yields a pattern similar to the fractal structure of the Poincare section for this system.

The resulting distribution function describes a stationary state independent of time. However, in some cases, for instance when studying a system's radiation spectra, a more complete description is required, necessitating knowledge of the distribution function's time dependence. This situation is similar to the description of a gas in statistical physics using a distribution function. In the equilibrium state, the distribution function is time-independent; however, for a more detailed description, it is necessary to account for the fluctuations of the distribution function, which are determined by its dependence on time (Lifshitz and Pitaevskii 1981). In this paper, we derive an equation for a smoothed distribution function that describes the equilibrium chaotic state of a dynamical system and whose time dependence allows for the description of fluctuations in the distribution function. The obtained results are illustrated by numerical calculations of the radiation intensity for a number of systems.

The equation for the smoothed distribution function

The dynamics equation for a nonlinear system may be written as:

$$\dot{\mathbf{r}} = \mathbf{f}(\mathbf{r}), \tag{1}$$

where $\mathbf{r}$ and $\mathbf{f}$ are vectors in the n–dimensional phase space of the system, and the time derivative is on the left side of the equation. An autonomous system of equations is considered, that is, the function $\mathbf{f}$ does not explicitly depend on t. A chaotic state in such systems is possible at $n > 2$. We will consider a dissipative system for which:

$$\mathrm{div}(\mathbf{f}) < 0.$$

In this case, for a certain set of parameters, the solution of the equation may be chaotic, and the attractor of such a system is a chaotic attractor (a strange attractor by Lorentz's definition).

In the paper by Liapzev (Liapzev 2019), it is shown for a special case that when system (1) describes the dynamics of a rotator in an external harmonic field, using the limit transition from the corresponding quantum mechanical problem, an equation for the probability density can be derived, which in the limit t takes the form:

$$\mathbf{f} \cdot \mathrm{grad}(\rho(\mathbf{r})) = 0, \tag{2}$$

where the dot represents the scalar product of the vectors.

In Kuznetsov's textbook (Kuznetsov 2006), when considering the chaotic state of a system of the form (1), the equation for the distribution function is derived as:

$$\dot{\rho}(\mathbf{r},t) + \mathrm{div}(\mathbf{f}\rho(\mathbf{r},t)) = 0. \tag{3}$$

As noted by the author, for conservative systems this equation is known as the Liouville equation. The derivation used by (Kuznetsov 2006) follows the standard approach found in textbooks on statistical physics (see, for example, (Kuni 1981)). According to this derivation, the distribution function at time t is related to the distribution function $\rho(\mathbf{r}_0, 0)) = 0$ at time $t = 0$ by the relation:

$$\rho(\mathbf{r},t) = \int \delta(\mathbf{r} - \mathbf{R}(t,\mathbf{r}_0))\rho(\mathbf{r}_0,0)d\mathbf{v}_0. \tag{4}$$

In this expression, $\mathbf{r}$ is a vector in phase space, $\mathbf{R}(t,\mathbf{r}_0)$ is the solution of equation (1) such that $\mathbf{R}(0,\mathbf{r}_0) = \mathbf{r}_0$ and integration is performed over the entire phase space.

For conservative systems, assuming that there is a certain limit at $t\to\infty$, so that $\dot{\rho}(\mathbf{r},t) = 0$, equation (3) reduces to equation (2). However, for dissipative systems, which are not conservative, equation (3) takes the form:

$$\dot{\rho}(\mathbf{r},t) + \mathrm{div}(\mathbf{f})\rho(\mathbf{r},t) + \mathbf{f}\cdot \mathrm{grad}(\rho(\mathbf{r},t)) = 0 \tag{5}$$

and for $\dot{\rho}(\mathbf{r},t) = 0$, it no longer reduces to equation (2).

To understand the discrepancy between equation (2) and equation (5) for $\dot{\rho}(\mathbf{r},t) = 0$, note that the theory of first-order partial differential equations (Kamke 1967) offers methods by which an inhomogeneous equation (an equation containing not only derivatives of ρ, but also the function itself) can be reduced to a homogeneous equation. This can be demonstrated most simply for the case $\mathrm{div}(\mathbf{f}) = -\gamma$, where γ is a positive constant. Note that this case is realized in many problems of nonlinear dynamics, particularly, in the Lorentz system where the chaotic (strange) attractor was first identified. Namely, the equation for the function $w(\mathbf{r},t) = \rho(\mathbf{r},t)\exp(-\gamma t)$ has the form:

$$\dot{w}(\mathbf{r},t) + \mathbf{f}\cdot \mathrm{grad}(w(\mathbf{r},t)) = 0. \tag{6}$$

Now, assume we have found a solution to equation (5). If the vector function $\mathbf{f}$ has no singularities, we can expect the function $w(\mathbf{r},t)$ to be bounded for all values of t. However, it follows that the maximum of the function $\rho(\mathbf{r},t) = w(\mathbf{r},t)\exp(\gamma t)$ will increase infinitely with time. Yet, as follows from equation (4), the norm of the function $\rho(\mathbf{r},t)$ remains constant; this means the region of phase space in which $\rho(\mathbf{r},t)$ is non-zero continuously shrinks over time

This can be most clearly demonstrated by considering the simplest one-dimensional system described by the equation:

$$\dot{x} = -\gamma x. \tag{7}$$

The solution of this equation has the form:

$$x(t) = x_0\exp(-\gamma t).$$

In accordance with expression (4), the distribution function $\rho(x,t)$ can be represented as an integral of the δ-function:

$$\rho(x, t) = \int \delta(x - x_0 e^{-\gamma t})\, \rho(x_0,0)dx_0, \tag{8}$$

where $\rho(x_0,0)$ is the distribution function at the initial time. The integral in the expression (8) is easily calculated, resulting in the expression:

$$\rho(x, t) = e^{\gamma t}\rho(e^{\gamma t}x,0).$$

Since the norm of the function is constant, the limit of the function $\rho(x,t)$ when $t\to\infty$ is a δ-function:

$$\lim_{t\to\infty}\rho\left(x,t\right) = \delta\left(x\right).$$

Thus, for the simplest one-dimensional dissipative system, as time tends to infinity, the distribution function tends to a δ-function. This corresponds to the fact that the attractor of the system described by equation (7) is a point in phase space.

For a dissipative system with more than one variable, other types of attractors appear (see, for example, (Grinchenko et al. 2007)). In particular, for a two-dimensional system, in addition to a point attractor, an attractor in the form of a limit cycle appears, which also takes the form of a δ-function:

$$\rho(\mathbf{r},t) = \delta(\mathbf{r} - \mathbf{R}(t)), \tag{9}$$

where $\mathbf{R}(t)$ is the periodic solution of equation (1) to which any solution of equation (1) tends, regardless of the initial conditions. If we attempt to represent a distribution function of the form (9) in phase space, it will exactly repeat the line $\mathbf{R}(t)$.

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In systems described by three or more variables, another type of attractor appears — the chaotic or strange attractor. In essence, we can assume that in this case the expression for the distribution function of the form (9) is also valid, only the period becomes infinitely long. As with an attractor that is a limit cycle, a distribution function of the form (9) can be represented as a line in a multidimensional space (see, for example, the Lorentz attractor).

Thus, the definition of a distribution function of the form (4) for dissipative systems with t tending to infinity provides little new information compared to the solution of equation (1). Secondly, it is completely unsuitable for numerical calculations that involve a distribution function, since numerical calculations always result in 'roughing' of data due to finite spatial cell sizes. For numerical calculations, it is advisable to somewhat 'smooth' the distribution function to obtain a continuous distribution function rather than the δ-function obtained for large times.

The simplest method to smooth the distribution function is to average it over small volumes of phase space, each surrounding a certain point. Mathematically, this can be described as follows. Let us choose a sufficiently large time interval T and consider the solution of equation (1) over the interval $[t, t+T]$. Consider an n-dimensional cube with a small edge l, centered at the vector $\mathbf{r}$. In accordance with the previous arguments, this cube is pierced by a large number of lines corresponding to δ-functions of the form (9). Let us denote the distribution function corresponding to this set of lines by $\rho(\mathbf{r},t,t+T)$. We now define the smoothed distribution function as the average of the distribution function by the expression:

$$\bar{\rho}(\mathbf{r},t) = \lim_{T\to\infty} \frac{C(t,T)}{l^n} \int d\mathbf{v}' \int_t^{t+T} dt' \rho(\mathbf{r}',t'), \tag{10}$$

where integration is performed over the volume of the n-dimensional cube with edge l, and $C(t,T)$ normalizes the smoothed distribution function. In fact, integration is performed along all lines corresponding to the δ-functions $\rho(\mathbf{r},t) = \delta(\mathbf{r} - \mathbf{R}(t))$ passing through a given n-dimensional cube.

Some qualitative consequences, as will be shown later, can be achieved by tending l to zero, while numerical calculations can be performed for finite small values of l.

We show that the equation for the smoothed distribution function has a form similar to equation (5), but without the term proportional to div($\mathbf{f}$):

$$\dot{\bar{\rho}}(\mathbf{r},t) + \mathbf{f} \cdot \mathrm{grad}(\bar{\rho}(\mathbf{r},t)) = 0. \tag{11}$$

To demonstrate this, consider the simplest case of a three-dimensional phase space, in which a strange attractor may already appear.

Consider a point with coordinates $\mathbf{r}_0$ and denote the corresponding vector $\mathbf{f}_0 = \mathbf{f}(\mathbf{r}_0)$. Let us construct a cube with an edge of size l and select the axes of a local coordinate system (x',y',z'), with its origin at the point $\mathbf{r}_0$, as shown in Fig. 1.

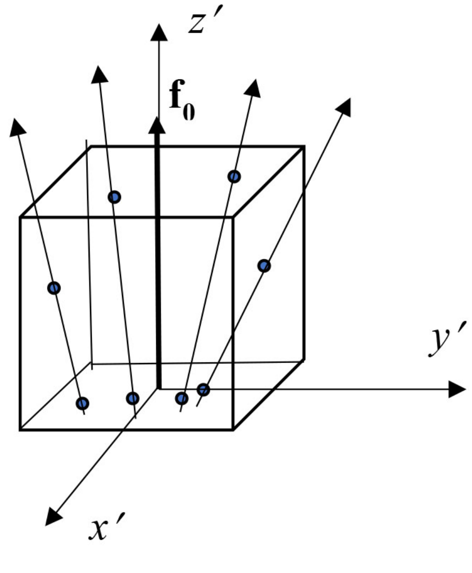

Fig. 1. Diagram for the derivation of equation (11)

Assuming the dimensions of the cube are small and the vector function **f**(**r**) is continuous, we expand it in a series in the vicinity of the cube:

$$\mathbf{f}(\mathbf{r}) = \mathbf{f}_0 + \mathbf{A}\mathbf{r}, \tag{12}$$

where the matrix is defined $\mathbf{A} = \begin{pmatrix} a_{x'x'} & a_{x'y'} & a_{x'z'} \\ a_{y'x'} & a_{y'y'} & a_{y'z'} \\ a_{z'x'} & a_{z'y'} & a_{z'z'} \end{pmatrix}$ and **r** is a vector with components x',y',z'. We will consider the edge of the cube to be so small that:

$$la \ll f_0 \,, \tag{13}$$

where $a = \max(|a_{ij}|)$. Thus, one can enter a small parameter:

$$\varepsilon = la/f_0,$$

which tends to zero as the size of the selected cube tends to zero.

The lines corresponding to the δ-functions (4) run along the trajectories **f**(**r**), which, according to the decomposition (12) and inequality (13), are straight lines running at small angles of the order ε to the z' axis, and penetrate the cube. Examples of these lines are shown in Fig. 1, where the dots mark the intersections of the lines with the faces of the cube. The figure shows the case of a divergent bundle of lines, corresponding to positive values of the parameters of the matrix **A**. For simplicity, we will consider this case further.

Let us now perform an averaging of the form (10) for the second term of equation (3), using the divergence theorem:

$$\int \mathrm{div}(\mathbf{f}(\mathbf{r}')\rho(\mathbf{r}',t,t+T))d\mathbf{v}' = \int \mathbf{f}(\mathbf{r}')\rho(\mathbf{r}',t,t+T)d\mathbf{s}, \tag{14}$$

where the integration on the right side of the equality occurs over the surface of the cube. Integrals with δ-functions are calculated trivially.

In accordance with Fig. 1, we distinguish between the lower and upper bases of the cube and its side surfaces. We will show that the flux through the side surfaces is of a higher order with respect to the parameter ε. Limiting the calculation to terms of no higher than the first order in ε, the flux through the lower and upper bases can be represented as:

$$\Phi(0) = -CN(0)f_{z'}(0), \quad \Phi(l) = CN(l)f_{z'}(l), \tag{15}$$

where N is the number of lines passing through the lower and upper bases, $f_{z'}$ is the average value of the projection of the vector **f** onto the z' axis on the corresponding base, and C is a normalization constant. The total flux (the right side of the equation (14)) is the sum of the fluxes (15). Let us now introduce the notation: $\Delta N = N(l) - N(0)$ and $\Delta f_{z'} = f_{z'}(l) - f_{z'}(0)$. Neglecting higher-order terms, the flux is given by the expression:

$$\Phi = Cf_0\Delta N - CN(0)\Delta f_{z'}\,. \tag{16}$$

It is readily understood that the first term in expression (16) characterizes a change in the distribution function ρ and corresponds to the term proportional to grad($\rho(\mathbf{r},t)$) in expression (5), while the second term in expression (16) characterizes the change in the vector **f** and corresponds to the term proportional to div(**f**) in expression (5). We denote these accordingly as:

$$\Phi_f = -CN(0)\Delta f_{z'}\,, \quad \Phi_\rho = Cf_0\Delta N.$$

Let us now show that for $l \to 0$, the ratio $\Phi_f/\Phi_\rho \to 0$. The value $\Delta f_{z'}$ is of the first order in ε. Thus:

$$\left|\Phi_f / \Phi_\rho\right| = \varepsilon \frac{N(0)}{\Delta N}.$$

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The number of lines crossing a given surface area obviously increases with the size of that area. For some continuous function, such as **f**, this number is proportional to the area for small surfaces. However, in this case, as $T \to \infty$, the set of points corresponding to the δ-functions (9) (the Poincare section for a chaotic attractor) is a fractal for which, for small area values, the relation holds:

$$\frac{N(0)}{\Delta N} = \left(\frac{S(0)}{\Delta S}\right)^d, \tag{17}$$

where $d < 1$, $S(0)$ is the area of the base of the cube, and ΔS is the area of the shaded shape shown in Fig. 2. The lines entering the cube through this shape exit through the side faces.

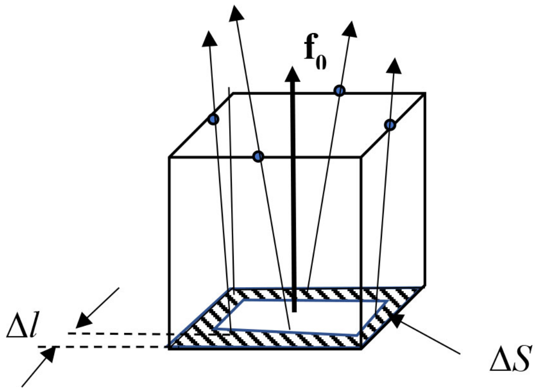

Fig. 2. Diagram for the derivation of equation (18)

Since the lines $\mathbf{f}(\mathbf{r})$ are directed at small angles of order ε to the vector $\mathbf{f}_0$, the figure with area ΔS is a strip of width approximately $\Delta l = \varepsilon l$ and length $4l$. Substituting these values into formula (17), we obtain:

$$\frac{N(0)}{\Delta N} \approx \left(\frac{1}{4\varepsilon}\right)^d.$$

As a result, for the ratio of fluxes Φ_f/Φ_ρ, the expression is obtained:

$$\left|\Phi_f / \Phi_\rho\right| \approx \frac{1}{4^d}\varepsilon^{1-d}. \tag{18}$$

Since $d < 1$, expression (18) implies that for $l \to 0$, the ratio $\Phi_f/\Phi_\rho \to 0$. Note that the flux through the side surfaces is proportional to ΔN, but is of order ε times smaller than the flux Φ_ρ. Thus, it can be concluded that when smoothing the distribution function, the term proportional to div(**f**) in expression (5) becomes negligible compared to the term proportional to grad(ρ). Note that if the function $\rho(\mathbf{r},t)$ were continuous, the value of d would be 1, and the fluxes Φ_f and Φ_ρ would be of the same order of smallness.

This conclusion can be generalized to a space with dimension greater than 3 by making the general assumption that when averaging over a small volume in phase space, the derivatives of the continuous function $\mathbf{f}(\mathbf{r})$ decrease faster than the derivatives of the function $\rho(\mathbf{r},t)$, which is not continuous as $t \to \infty$.

Calculation of the radiation intensity spectrum through the fluctuation spectrum of a smoothed distribution function

In systems treated by the methods of statistical physics, the time evolution of the average statistical value of a quantity G, expressible in terms of the system variables **r**, is calculated by averaging according to the general rules (see, for example, (Kuni 1981)):

$$\langle G(t)\rangle = \int G(\mathbf{r})\rho(\mathbf{r},t)d\mathbf{v}. \tag{19}$$

When considering the evolution of a state near equilibrium, the distribution function can be represented as:

$$\rho(\mathbf{r},t) = \rho_0(\mathbf{r}) + \delta\rho(\mathbf{r},t),$$

where $\rho_0(\mathbf{r})$ is the equilibrium distribution function, and $\delta\rho(\mathbf{r},t)$ describes small fluctuations near equilibrium. Accordingly, from formula (19) we obtain:

$$\langle G(t)\rangle = G_0 + \langle \delta G(t)\rangle,$$

where the value $\langle \delta G(t)\rangle$ is determined by the fluctuation of the distribution function $\langle \delta G(t)\rangle$. Note that the values of the function $\langle \delta G(t)\rangle$ are not necessarily small compared to G_0, since, for example, G_0 can be zero due to symmetry.

In many cases, the primary interest is not the time dependence of $\langle \delta G(t)\rangle$, but its Fourier transform.

$$\langle \delta G_\omega \rangle = \int_{-\infty}^{\infty} \delta G(t) e^{i\omega t} dt .$$

Accordingly, the values of δG_ω are determined by the Fourier transform of the distribution function fluctuations:

$$\delta\rho_\omega(r) = \int_{-\infty}^{\infty} \delta\rho(\mathbf{r},t) e^{i\omega t} dt .$$

Note that for conservative systems, where div($\mathbf{f}$) = 0, equation (5) is similar to the equation obtained above for the smoothed distribution function (11). The similarity, however, is not complete. In both cases, the equation for the distribution function has the form:

$$\dot{\rho}(\mathbf{r},t) + iL\rho(\mathbf{r},t) = 0,$$

where the operator L is defined by the equality:

$$L\rho(\mathbf{r},t) = -i\mathbf{f} \cdot \mathrm{grad}(\rho(\mathbf{r},t)).$$

However, while for conservative systems the Liouville operator L is self-adjoint (see (Kuni 1981)), for dissipative systems it is not. Nevertheless, we can attempt to use the calculation of distribution function fluctuations to determine the spectra of physical quantities in the equations of the nonlinear dynamics for systems in a chaotic state. In particular, the emission spectra of a dynamical system are determined by the Fourier transform of the time derivative of the system's dipole moment, denoted here by V:

$$I(\omega) = \omega^2 |V_\omega|^2. \quad (20)$$

The Fourier transform V_ω can be determined directly by solving the system of equations (1):

$$V_\omega = \int_0^T V(\mathbf{R}(t)) e^{i\omega t} dt , \quad (21)$$

where $\mathbf{R}(t)$ is the solution of the system of equations (1) obtained over the interval from 0 to T. The applicability of the smoothed distribution function can be tested by comparing calculations using formulas (20) and (21) with calculations that determine fluctuations in the smoothed distribution function. The corresponding expressions are:

$$\langle I_\omega \rangle = \omega^2 |\langle V_\omega \rangle|^2, \quad (22)$$

$$\langle V_\omega \rangle = \int V(\mathbf{r}) \bar{\rho}_\omega(\mathbf{r}) d\mathbf{v}, \quad (23)$$

where $\bar{\rho}_\omega(\mathbf{r})$ p is the Fourier transform of the smoothed distribution function.

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Formulas (21) and (23) simplify in the case when the variable V is one of the variables in phase space, so that:

$$\mathbf{r} = (V, r_2, r_3, \ldots r_n),$$

Formula (21) then transforms to:

$$V_\omega = \int_0^T V(t) e^{i\omega t} dt .$$

For the average value $\langle V_\omega \rangle$, we get:

$$\langle V_\omega \rangle = \int F_\omega(V) V dV,$$

where the function

$$F_\omega(V) = \int dr_2 dr_3 \ldots dr_n \bar{\rho}_\omega(V, r_2, r_3, \ldots r_n)$$

can be considered the Fourier transform of the distribution function over the variable V.

Numerical calculations

The feasibility of calculating the radiation spectra intensity of mechanical systems by first computing the fluctuations of the smoothed distribution function was tested on two mechanical systems where chaotic oscillations occur for specific parameter sets.

The first system chosen was an oscillator with a W-shaped potential under external harmonic forces, known as the Duffing oscillator. As a second example, self-oscillations in a mechanical system with two degrees of freedom under the influence of external dry friction force were considered (Liaptsev 2010).

The calculation of the function $F_\omega(V)$ was performed by solving the system of equations describing the system's dynamics (equations (1)) over a time interval from 0 to $T_{\max}$. This interval was divided into n_t-1 equal subintervals. The variable ω took n_ω equidistant discrete values in the range from 0 to $\omega_{\max}$. The variable V took n_v equidistant discrete values in the range from $V_{\min}$ to $V_{\max}$. By solving the system of differential equations, the value $V_{mk} \approx V(t_i)\exp(i\omega_m t_j)$ was determined for each time t_j. Summing these values determined the array $F_{mk} = F_{\omega_m}(V_k)$. Normalization was performed at the value ω_1=0:

$$\sum_{k=1}^{n_v} F_{mk} = 1 .$$

The Duffing oscillator

The Duffing equation, which describes, in particular, forced oscillations of an oscillator with a W potential, can be reduced via large-scale transformations to the form:

$$\ddot{x} + \gamma\dot{x} + \alpha x + \beta x^3 = f\cos(\Omega t).$$

This non-autonomous second-order differential equation can be reduced to an autonomous system of equations of the form (1):

$$\begin{aligned} &\dot{x} = V \\ &\dot{V} = -\gamma V - \alpha x - \beta x^3 + f\cos\varphi. \\ &\dot{\varphi} = \Omega \end{aligned}$$

A chaotic solution, for which the calculation was performed, occurs with the following parameter set: $\alpha = -1, \rightarrow \beta = 1, \gamma = 0.3, f = 0.4, \Omega = 1.2$. The parameters used in the numerical calculations were: $V_{\max} = -V_{\min} = 0.97$, $n_v = 101$, $\omega_{\max} = 6$, $n_\omega 100$, $T_{\max} = 2000\pi/\Omega$, n_t-106.

The results of the intensity calculation are shown in Fig. 3.

In the figure, the dots represent the values calculated by formula (20), and the circles represent the values calculated by formula (22).

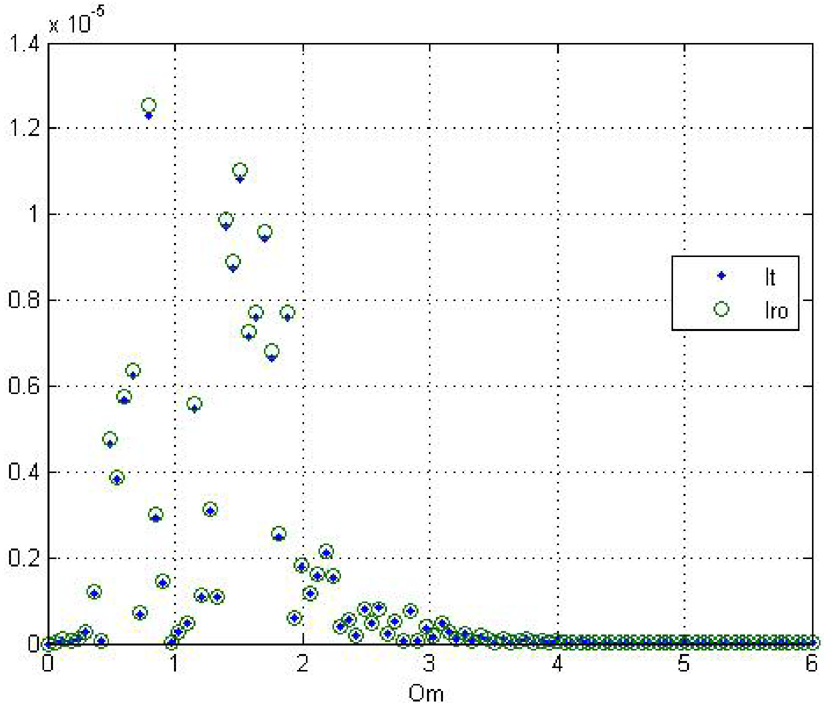

Fig. 3. Calculation results for the Duffing oscillator

As can be seen from the figure, reasonably good agreement between the calculations is obtained, indicating that the description of chaotic states using a smoothed distribution function is adequate. The figure also shows the chaotic nature of the spectrum. For other parameters, for instance, reducing the value of f to 0.37 while keeping the others constant, results in a regular periodic motion (an attractor in the form of a limit cycle) instead of a chaotic state. Accordingly, the radiation spectrum acquires a structure of distinct narrow lines. However, as calculations show, using the smoothed distribution function still yields results consistent with the direct spectrum calculation (20).

Chaotic self-oscillations under the influence of dry friction force

Mechanical self-oscillations under the influence of dry friction force occur, for example, when a bow moves along a string. The system shown in Fig. 4 (Kondratyev and Liaptsev 2008) is often considered as a model problem for studying such self-oscillations.

The system is a body on a moving conveyor belt, connected by elastic force to a fixed support. Self-oscillations arise because the dry friction force depends on the velocity of the body relative to the belt, the modulus of which decreases with increasing modulus of velocity (Fig. 5).

The equation of motion for this system reduces to a system of equations (1). However, since the phase space dimension in this case is 2, the attractor is a limit cycle. The possibility of chaotic self-oscillations arises in systems with a greater number of degrees of freedom. A system proposed in (Liaptsev 2010) is shown in Fig. 6.

The device resembles a chart recorder, where a pen slides along a moving tape, leaving a trace. A body slides along a horizontal tape moving at speed u (point A in the figure). The body is fixed to a rod, which

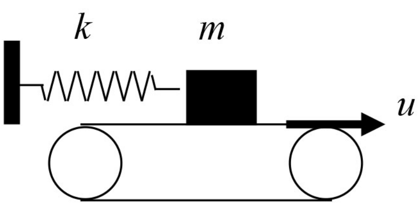

Fig. 4. A model for studying self-oscillations under the influence of dry friction force

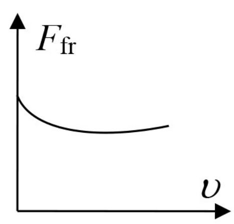

Fig. 5. Dependence of the modulus of dry friction force on velocity

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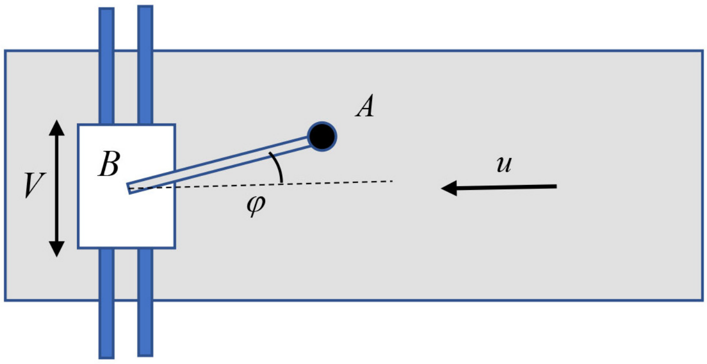

Fig. 6. A model demonstrating the occurrence of chaotic self-oscillations under the influence of dry friction force

can rotate so that the angle φ between the rod and the tape can vary within certain limits. These limits are determined by a nonlinear elastic force that returns the rod to its equilibrium position ($\varphi = 0$). A second degree of freedom is introduced by a 'carriage' (body B) on which the rod is fixed; this carriage can move translationally in a direction perpendicular to the tape along a set of guides. Accordingly, the carriage speed V may vary within certain limits.

The equations governing the motion of the system contain several parameters, and their detailed derivation is described in (Liaptsev 2010). A key control parameter is the tape speed u. Depending on this parameter, the resulting self-oscillations can be either regular (periodic) or chaotic.

In this work, the spectrum of chaotic self-oscillations for such systems is determined using the methods described above (Fig. 7). The dipole moment of the system is assumed to be determined by the position of carriage B. The frequency of small free vibrations of the rod under the action of the elastic force is chosen as the unit frequency. The graphs show the values of I/ω^2. As in the previous example, the dots indicate values calculated by formula (20), and the circles indicate the values calculated by formula (22).

As in the previous example, the chaotic nature of the spectrum is evident, and good agreement is observed between the spectrum calculated by the direct method (formula (20)) and that calculated using the smoothed distribution function (formula (22)).

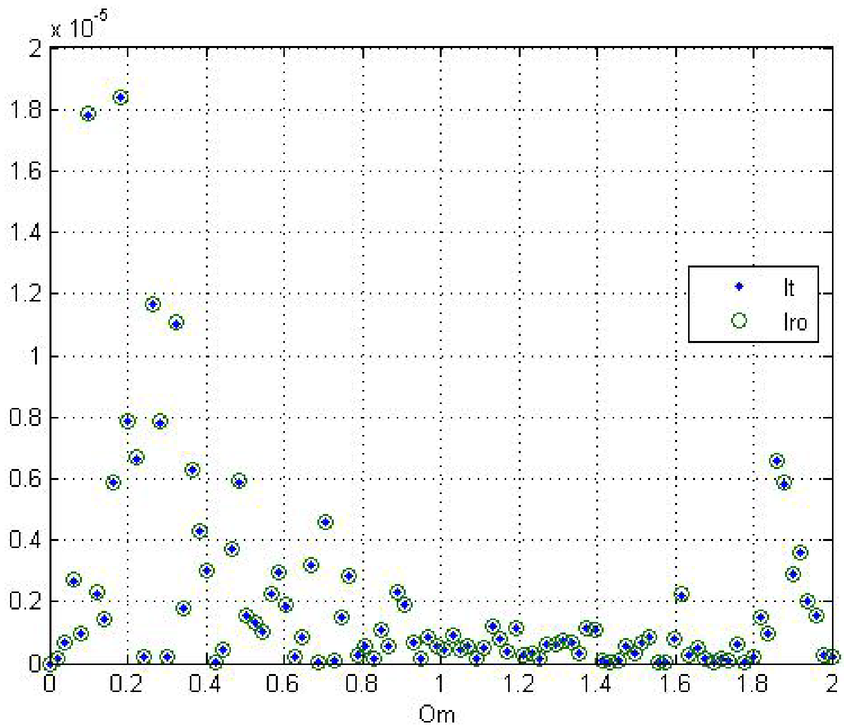

Fig. 7. Results of the spectrum calculation for a model demonstrating chaotic self-oscillations under the action of dry friction

Conclusion

The numerical calculations presented here are, of course, of a private nature. However, they confirm the feasibility of using a smoothed distribution function to describe the chaotic states of systems governed by nonlinear dynamic equations. Unlike the exact distribution function defined in the methods of statistical physics, the smoothed distribution function in these problems is continuous and more consistent with numerical calculations performed for specific systems.

Conflict of Interest

The authors declare that there is no conflict of interest, either existing or potential.

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