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Potential for a transition from standard to phantom dark energy in the FLRW cosmology: A dynamical systems analysis

V. D. Vertogradov [✉1,2,3], U. V. Yamaltdinova [2]

[1] Herzen State Pedagogical University of Russia, 48 Moika Emb., Saint Petersburg 191186, Russia
[2] Saint Peterburg Branch of Special Astrophysical Observatory of the Russian Academy of Sciences, 65 Pulkovskoe Highway, Saint Petersburg 196140, Russia
[3] Center for Theoretical Physics, Khazar University, 41 Mehseti Street, Baku AZ-1096, Azerbaijan

Authors

Vitaly D. Vertogradov, ORCID: 0000-0002-5096-7696, e-mail vdvertogradov@gmail.com

Ulyana V. Yamaltdinova, ORCID: 0009-0001-8451-0950, e-mail ulanav498@gmail.com

Abstract. Modern cosmology is based on the ΛCDM model, in which the Universe's accelerated expansion is driven by dark energy with equation of state $w = -1$. While ΛCDM agrees well with observational data — from CMB anisotropies to large-scale structure — both theoretical considerations and emerging observations suggest that dark energy may be dynamical, with $w(z)$ that evolves over time. A scenario of particular interest is a potential transition from standard dark energy ($w \geq -1$) to phantom energy ($w < -1$), which violates the strong energy condition and may lead to a 'Big Rip'. Planck data (Aghanim et al. 2020) confirm that, as of today, dark energy dominates ($\Omega_{DE} \approx 0.685$) and w is consistent with -1; however, combined analyses (Planck + BAO + supernovae) permit slight deviations into the phantom regime ($w \approx -1.03 \pm 0.03$). Though marginal, this possibility motivates the study of models where $w(z)$ crosses the $w = -1$ barrier. Moreover, recent JWST observations reveal unexpectedly massive galaxies at $z > 10$, challenging ΛCDM predictions of structure formation and hinting at modified expansion histories that may involve evolving — or phantom — dark energy. This work investigates whether phantom energy can act as a future attractor in a flat FLRW cosmology. Using autonomous dynamical systems, we analyse the evolution of density parameters $\Omega_i(z)$ and $w(z)$ in a model that permits crossing the $w = -1$ divide.

Keywords: cosmology, dark energy, phantom energy, FLRW model, dynamical systems, equation of state

Einstein's field equations

In 1905, Albert Einstein formulated the special theory of relativity (STR), which unified space and time into a single four-dimensional spacetime and rejected Newton's concept of absolute space and time (Blau 2024). However, STR is restricted to inertial reference frames and does not incorporate gravity.

Seeking to include gravitation into the relativistic worldview, Einstein completed the general theory of relativity (GTR) in 1915 — a geometric theory in which gravitational interaction is interpreted not as a force, but as a manifestation of spacetime curvature induced by the distribution of energy and

momentum (Blau 2024). The foundation of the theory is given by Einstein's equations, which in their modern form read:

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu},$$ (1)

where $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} R\, g_{\mu\nu}$ is the Einstein tensor, expressing spacetime curvature via the Ricci tensor $R_{\mu\nu}$ and Ricci scalar R; $g_{\mu\nu}$ is the metric tensor, defining intervals and angles; $T_{\mu\nu}$ is the energy–momentum tensor, describing the densities and fluxes of energy, momentum, and stresses; G is the gravitational constant, c is the speed of light, and Λ is the so-called *cosmological constant* (Adler et al.1995; Blau 2024).

A crucial feature of the left-hand side is that, owing to the Bianchi differential identity ($\nabla^\mu G_{\mu\nu} = 0$) and metric compatibility ($\nabla^\mu g_{\mu\nu} = 0$), Einstein's equations automatically imply the law of local energy–momentum conservation:

$$\nabla^\mu T_{\mu\nu} = 0.$$ (2)

This ensures that GTR remains consistent with fundamental physical principles, even within curved spacetime (Perko 2001).

As early as 1916, Karl Schwarzschild found the first exact vacuum solution ($T_{\mu\nu} = 0$) to Einstein's equations, describing spacetime around a spherically symmetric body — a result that later became the cornerstone for the theory of black holes (Blau 2024). However, as it turned out, local solutions were only a first step: to describe the Universe as a whole, a solution satisfying the new physical principle of large-scale homogeneity and isotropy was required (Coley 2003).

A significant difficulty then arose: the original field equations (with $\Lambda = 0$) admit no static, homogeneous, and isotropic solution — any such model is necessarily dynamical, either expanding or contracting. This conflicted with the early-20th-century view of a static, eternal Universe (Weinberg 1989).

To resolve this inconsistency, Einstein in 1917 introduced the additional term $\Lambda g_{\mu\nu}$ into the field equations. Initially, Λ had no physical interpretation — it was a *geometric parameter* introduced *ad hoc* to allow for a static cosmological solution. In his paper 'Cosmological Considerations in the General Theory of Relativity', he showed that the condition

$$\Lambda = \frac{4\pi G\rho}{c^2},$$ (3)

where ρ is the average mass density in a homogeneous Universe, yields a static model (later termed *Einstein's Universe* — a closed three-dimensional hypersphere of finite radius). Thus, Λ functioned as *gravitational repulsion*, precisely compensating the attraction of matter (Adler et al. 1995; Weinberg 1989).

Importantly, the term $\Lambda g_{\mu\nu}$ was mathematically permissible — it preserved general covariance and automatically satisfied $\nabla^\mu(\Lambda g_{\mu\nu}) = 0$, thereby not violating energy–momentum conservation (Blau 2024). However, in 1922–1929, A. A. Friedmann, G. Lemaître, and E. Hubble demonstrated the Universe's expansion, rendering the static model obsolete. Einstein soon abandoned Λ, calling it 'the greatest blunder' of his life.

Paradoxically, observations of the Universe's accelerated expansion from Type Ia supernovae data in 1998 revived Λ as central to modern cosmology (Dungan, Prosper 2011; Riess et al. 1998). It is now interpreted as the simplest model of *dark energy* — a homogeneous vacuum energy with an equation of state $p = -\rho c^2$ (Carroll 2001; Peebles, Ratra 2003; Weinberg 1989). Thus, the parameter introduced to salvage an outdated cosmological picture proved essential for describing the observed accelerated expansion. It is with this modern interpretation that Λ enters the subsequent stage: constructing a dynamical cosmological model.

The Friedmann metric

The expansion of the Universe, discovered by Hubble, necessitated a transition from static to dynamical cosmological models (Aghanim et al. 2020). Already in the 1920s, A. A. Friedmann, and independently G. Lemaître, H. Robertson, and A. Walker, showed that assuming the *cosmological principle* — homogeneity and isotropy of the Universe on scales $\gtrsim 100$ Mpc — uniquely determines spacetime geography via the Friedmann–Robertson–Walker (FRW) metric (Coley 2003).

Although the Universe is structured on small scales (galaxies, clusters, filaments, and voids), averaging over sufficiently large volumes allows its description as a smooth medium (Barreiro et al. 2000; Nunes, Mimoso 2000). Based on this, the most general spacetime metric consistent with the cosmological principle is:

$$ds^2 = -c^2 dt^2 + a^2(t)[\frac{dr^2}{1 - kr^2} + r^2(d\theta^2 + \sin^2\theta\, d\phi^2)], \qquad (4)$$

where: ds^2 is the spacetime interval squared; t is *cosmic time*, measured by an observer comoving with the average matter flow; $a(t)$ is the *scale factor*, a dimensionless function of time that determines the relative change in distances between comoving points (e. g., unbound galaxies); (r, θ, ϕ) are comoving spherical coordinates ('attached' to the expanding medium); k is the dimensionless spatial curvature parameter, taking one of three values:

$$k = \begin{array}{ll} +1 & \text{— closed (spherical) space,} \\ 0 & \text{— flat (Euclidean) space,} \\ -1 & \text{— open (hyperbolic) space.} \end{array} \qquad (5)$$

The scale factor $a(t)$ itself has no absolute physical meaning — it may be normalised arbitrarily (e. g., $a(t_0) = 1$ at the present epoch). Observable quantities depend only on its *derivatives*. Hence, the key dynamical characteristic in cosmology is the *Hubble parameter*:

$$H(t) = \frac{\dot{a}(t)}{a(t)}, \qquad (6)$$

where the dot denotes differentiation with respect to cosmic time t. The quantity $H_0 = H(t_0)$ characterises the current expansion rate (Aghanim et al. 2020).
Consider the physical meaning of the three possible values of k:
● $k = +1$ (*closed Universe*). Spatial sections have positive curvature and the topology of a three-sphere S^3. The volume is finite and unbounded. If the total energy density ρ exceeds the *critical density*

$$\rho_{\text{cr}}(t) = \frac{3H^2(t)}{8\pi G}, \qquad (7)$$

gravitational attraction will eventually halt expansion, and the Universe will recollapse into a singularity — the 'Big Crunch' (Coley 2003).
● $k = 0$ (*flat Universe*). Space is Euclidean ($\mathbb{R}^3$) and infinite. In the absence of dark energy, expansion continues forever but gradually decelerates. Modern CMB data (*Planck*, 2018) indicate a high degree of flatness:

$$\Omega_k \equiv -\frac{kc^2}{a^2 H^2} = -0.011^{+0.010}_{-0.012}, \qquad (8)$$

consistent with $k = 0$ at the 1% level (Aghanim et al. 2020).
● $k = -1$ (*open Universe*). Space has negative curvature (Lobachevskian geometry), is infinite, and geodesics diverge. This model occurs when $\rho < \rho_{\text{cr}}$, and expansion proceeds indefinitely. The sum of angles in a triangle is less than 180 °, and the volume of a sphere of radius R grows faster than R^3 (Coley 2003).
Thus, the FRW metric is not an arbitrary choice, but the *only* form compatible with the cosmological principle. It provides the geometric 'stage' on which cosmic dynamics unfolds. To 'activate' these dynamics — that is, to determine how exactly $a(t)$ evolves with time — one must substitute the FRW metric into Einstein's equations and specify the material content of the Universe. This step leads directly to the system of equations governing cosmological evolution.

Dynamics of cosmic expansion: From Einstein's equations to the evolution of the scale factor

Our primary objective is to determine the time evolution of the scale factor $a(t)$ — i. e., to describe the dynamics of cosmic expansion (or contraction). To this end, we begin with the fundamental equations of general relativity — Einstein's field equations in their general form:

$$G_{\mu\nu} + \Lambda\, g_{\mu\nu} = \frac{8\pi G}{c^4}\, T_{\mu\nu}. \tag{9}$$

Here: $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} R\, g_{\mu\nu}$ is the Einstein tensor, expressing spacetime curvature; $g_{\mu\nu}$ is the metric tensor (in this context, the Friedmann metric); Λ is the cosmological constant, introduced to describe homogeneous vacuum energy; $T_{\mu\nu}$ is the energy–momentum tensor, describing the distribution and motion of matter and energy; G and c are fundamental constants (Blau 2024).

Since matter on cosmological scales can be modeled as a perfect fluid (dust, radiation, dark energy), we adopt

$$T_{\mu\nu} = (\rho + \frac{p}{c^2})u_\mu u_\nu + p\, g_{\mu\nu}, \tag{10}$$

where $\rho(t)$ is the energy density, $p(t)$ is the pressure, and u^μ is the four-velocity of a comoving observer (Coley 2003).

Substituting the FRW metric and this form of $T_{\mu\nu}$ into Eq. (9) — a technically cumbersome but straightforward procedure — yields two independent differential equations for $a(t)$, known as the *Friedmann equations*:

$$\left(\frac{\dot{a}}{a}\right)^2 = \left(\frac{8\pi G}{3}\right)\rho - \left(k\frac{c^2}{a^2}\right) + \left(\Lambda\frac{c^2}{3}\right), \tag{11}$$

$$\frac{\ddot{a}}{a} = -\left(\frac{4\pi G}{3}\right)\left(\rho + \frac{3p}{c^2}\right) + \frac{\Lambda c^2}{3}. \tag{12}$$

The first equation (often called the *Hubble equation*) links the current expansion rate to the total energy content and geometry of the Universe. The second equation shows that acceleration is determined not only by density but also by pressure: when $\rho + 3p/c^2 < 0$, gravity becomes effectively repulsive — this is the precise mechanism driving the accelerated expansion observed today (Copeland et al. 2006; Feng et al. 2005).

However, the system (11)–(12) contains three time-dependent unknowns: $a(t)$, $\rho(t)$, and $p(t)$. To obtain a unique solution, the system must be closed by an additional relation.

This relation emerges naturally from the structure of Einstein's equations. As noted earlier, the left-hand side of (9) is identically conserved: $\nabla^\mu(G_{\mu\nu} + \Lambda g_{\mu\nu}) = 0$, and therefore Einstein's equations imply the local energy–momentum conservation law:

$$\nabla_\mu T^{\mu\nu} = 0. \tag{13}$$

For a perfect fluid in the FRW metric, Eq. (13) reduces to a single nontrivial scalar equation — the *continuity equation*:

$$\dot{\rho} + 3H(\rho + \frac{p}{c^2}) = 0, H = \frac{\dot{a}}{a}. \tag{14}$$

Physically, it expresses energy balance in an expanding volume: as the scale factor grows ($a\uparrow$), the volume increases as a^3, thereby decreasing density; the pressure p determines whether additional energy loss occurs (e. g., photons lose energy not only due to dilution but also due to cosmological redshift) (Coley 2003).

Despite the presence of Eq. (14), the system remains underdetermined: ρ and p are still independent. To resolve this, we introduce an *equation of state* — a physical relation derived from the microscopic nature of the matter:

$$p = w\,\rho c^2, \tag{15}$$

where w is a dimensionless parameter characteristic of the given component:
- $w = 0$ — non-relativistic matter ('dust'): $\rho \propto a^{-3}$;
- $w = \dfrac{1}{3}$ — radiation: $\rho \propto a^{-4}$;
- $w = -1$ — cosmological constant: ρ = const;
- $w < -1$ — phantom energy: ρ *increases* with expansion (Cai et al. 2010; Garcia-Salcedo et al. 2013).

Substituting (15) into (14) yields the explicit dependence:

$$\rho(a) = \rho_0\,a^{-3(1+w)}, \tag{16}$$

which, when inserted into the first Friedmann equation (11), transforms it into a closed differential equation for $a(t)$. Thus, from fundamental Einstein equations — through geometry (FRW), matter ($T_{\mu\nu}$), and microphysics (w) — we arrive at a complete description of cosmic evolution.

Dynamical systems method in cosmology

Consider a homogeneous and isotropic Universe described by the Friedmann–Robertson–Walker (FRW) metric with zero spatial curvature ($k = 0$). The first Friedmann equation then takes the form:

$$\left(\frac{\dot{a}}{a}\right)^2 = H^2 = \frac{1}{3}\rho, \tag{17}$$

where $a(t)$ is the scale factor, $H \equiv \dot{a}/a$ is the Hubble parameter, and ρ is the total energy density. We assume the Universe consists of four components:
- radiation (r),
- non-relativistic matter (m),
- standard dark energy (DE),
- phantom energy (ph).

Consequently,

$$\rho = \rho_r + \rho_m + \rho_{\mathrm{DE}} + \rho_{\mathrm{ph}}. \tag{18}$$

The evolution of each component is governed by the continuity equation, derived from energy–momentum conservation in the FRW metric:

$$\dot{\rho}_i + 3H(\rho_i + p_i) = 0, \tag{19}$$

where p_i is the pressure of the i-th component. Introducing the equation-of-state parameter $w_i \equiv p_i/\rho_i$, Eq. (19) becomes

$$\dot{\rho}_i = -3H(1 + w_i)\rho_i. \tag{20}$$

For convenience, we define

$$\gamma_i \equiv 1 + w_i, \tag{21}$$

so that

$$\dot{\rho}_i = -3H\gamma_i\rho_i. \tag{22}$$

The specific values of γ_i for the components under consideration are determined by their physical nature:

$$\gamma_r = \frac{4}{3}, \gamma_m = 1, 0 \leq \gamma_{DE} < \frac{2}{3}, \gamma_{ph} < 0. \tag{23}$$

Direct integration of the system (17)–(22) is difficult, as $\rho_i(a)$ are governed by nonlinear differential equations. Therefore, for qualitative analysis, we employ the dynamical systems method. We introduce dimensionless density parameters

$$\Omega_i \equiv \frac{\rho_i}{3H^2}, i = r, m, DE, ph, \tag{24}$$

and substituting (18) into (17) yields the normalisation condition:

$$\Omega_r + \Omega_m + \Omega_{DE} + \Omega_{ph} = 1. \tag{25}$$

Thus, only three variables are independent. We choose as phase-space coordinates:

$$x \equiv \Omega_m, y \equiv \Omega_{DE}, z \equiv \Omega_{ph}, \tag{26}$$

from which it follows that

$$\Omega_r = 1 - x - y - z. \tag{27}$$

To obtain an autonomous system, we differentiate (26) with respect to the logarithm of the scale factor, $N = \ln a$ (noting that $d/dN = H^{-1}d/dt$). For example, for x:

$$\frac{dx}{dN} = \frac{1}{H}\frac{d}{dt}\left(\frac{\rho_m}{3H^2}\right) = \frac{1}{H}\left(\frac{\dot{\rho}_m}{3H^2} - \frac{2\rho_m\dot{H}}{3H^3}\right) = \frac{\dot{\rho}_m}{3H^3} - 2\frac{\rho_m\dot{H}}{3H^4}. \tag{28}$$

Substituting $\dot{\rho}_m = -3H\gamma_m\rho_m$ from (22) and expressing $\dot{H}$ via the second Friedmann equation:

$$\dot{H} = -\frac{1}{2}\sum_i(\rho_i + p_i) = -\frac{1}{2}\sum_i\gamma_i\rho_i, \tag{29}$$

and using $\rho_i = 3H^2\Omega_i$ and (27), we obtain:

$$\dot{H} = -\frac{3}{2}H^2[\gamma_r(1 - x - y - z) + \gamma_m x + \gamma_{DE}y + \gamma_{ph}z]. \tag{30}$$

Insertion of (22) and (30) into (28), together with $\gamma_m = 1$, $\gamma_r = 4/3$, yields after simplification:

$$\frac{dx}{dN} = x[1 - x + y(3\gamma_{DE} - 4) + z(3\gamma_{ph} - 4)]. \tag{31}$$

Analogously, the equations for y and z are:

$$\frac{dy}{dN} = y[4 - 3\gamma_{DE} + x + y(3\gamma_{DE} - 4) + z(3\gamma_{ph} - 4)], \tag{32}$$

$$\frac{dz}{dN} = z[4 - 3\gamma_{ph} + x + y(3\gamma_{DE} - 4) + z(3\gamma_{ph} - 4)]. \tag{33}$$

The system (31)–(33) is autonomous (right-hand sides depend only on x, y, z), allowing the application of dynamical systems theory (Boehmer, Chan 2014; Boehmer et al. 2012; Coley 2003; Perko 2001).

Critical points and the Jacobian matrix

Critical points are obtained from $\dot{x} = \dot{y} = \dot{z} = 0$. Nontrivial solutions yield four physically meaningful points:

$$
\begin{aligned}
\mathbf{P}_r &= (0,0,0) &&\text{— radiation domination,} \\
\mathbf{P}_m &= (1,0,0) &&\text{— matter domination,} \\
\mathbf{P}_{\mathrm{DE}} &= (0,1,0) &&\text{— standard dark energy domination,} \\
\mathbf{P}_{\mathrm{ph}} &= (0,0,1) &&\text{— phantom energy domination.}
\end{aligned}
\tag{34}
$$

To analyse their stability, we compute the Jacobian matrix $J = (\partial f_i / \partial x_j)$, where $f_1 = dx/dN$, etc. Denoting for brevity $A = 3\gamma_{\mathrm{DE}} - 4$, $B = 3\gamma_{\mathrm{ph}} - 4$, $C = 4 - 3\gamma_{\mathrm{DE}}$, $D = 4 - 3\gamma_{\mathrm{ph}}$, we obtain:

$$
J(x,y,z) = \begin{pmatrix} 1 - 2x + Ay + Bz & Ax & Bx \\ y & C + x + 2Ay + Bz & By \\ z & Az & D + x + Ay + 2Bz \end{pmatrix}.
\tag{35}
$$

Substituting the coordinates of each critical point:

1. Point $\mathbf{P}_r = (0, 0, 0)$

$$
J(\mathbf{P}_r) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 4 - 3\gamma_{\mathrm{DE}} & 0 \\ 0 & 0 & 4 - 3\gamma_{\mathrm{ph}} \end{pmatrix}.
\tag{36}
$$

Eigenvalues: $\lambda_1 = 1$, $\lambda_2 = 4 - 3\gamma_{\mathrm{DE}}$, $\lambda_3 = 4 - 3\gamma_{\mathrm{ph}}$.
For $0 \le \gamma_{\mathrm{DE}} < 2/3$ and $\gamma_{\mathrm{ph}} < 0$, all $\lambda_i > 0$.
Conclusion: $\mathbf{P}_r$ is an unstable source (repeller), corresponding to the initial state.

2. Point $\mathbf{P}_m = (1, 0, 0)$

$$
J(\mathbf{P}_m) = \begin{pmatrix} -1 & 3\gamma_{\mathrm{DE}} - 4 & 3\gamma_{\mathrm{ph}} - 4 \\ 0 & 5 - 3\gamma_{\mathrm{DE}} & 0 \\ 0 & 0 & 5 - 3\gamma_{\mathrm{ph}} \end{pmatrix}.
\tag{37}
$$

Eigenvalues: $\lambda_1 = -1$, $\lambda_2 = 3(1 - \gamma_{\mathrm{DE}})$, $\lambda_3 = 3(1 - \gamma_{\mathrm{ph}})$.
For $\gamma_{\mathrm{DE}} < 1$ and $\gamma_{\mathrm{ph}} < 0$, we have $\lambda_1 < 0, \lambda_2 > 0, \lambda_3 > 0$.
Conclusion: $\mathbf{P}_m$ is a saddle point, corresponding to an intermediate epoch.

3. Point $\mathbf{P}_{\mathrm{DE}} = (0, 1, 0)$

$$
J(\mathbf{P}_{\mathrm{DE}}) = \begin{pmatrix} 3(\gamma_{\mathrm{DE}} - 1) & 0 & 3\gamma_{\mathrm{ph}} - 4 \\ 1 & 0 & 0 \\ 0 & 0 & 3(\gamma_{\mathrm{DE}} - \gamma_{\mathrm{ph}}) \end{pmatrix}.
\tag{38}
$$

Eigenvalues: $\lambda_1 = 3(\gamma_{\mathrm{DE}} - 1) < 0$, $\lambda_2 = 3\gamma_{\mathrm{DE}} - 4 <$ since $(\gamma_{\mathrm{DE}} < 2/3)$, $\lambda_3 = 3(\gamma_{\mathrm{DE}} - \gamma_{\mathrm{ph}}) > 0$ as $(\gamma_{\mathrm{ph}} < 0 < \gamma_{\mathrm{DE}})$.
Conclusion: $\mathbf{P}_{\mathrm{DE}}$ is a saddle point.

4. Point $\mathbf{P}_{\mathrm{ph}} = (0, 0, 1)$

$$
J(\mathbf{P}_{\mathrm{ph}}) = \begin{pmatrix} 3(\gamma_{\mathrm{ph}} - 1) & 3\gamma_{\mathrm{DE}} - 4 & 0 \\ 0 & 3(\gamma_{\mathrm{ph}} - \gamma_{\mathrm{DE}}) & 0 \\ 1 & 3\gamma_{\mathrm{DE}} - 4 & 0 \end{pmatrix}.
\tag{39}
$$

Eigenvalues: $\lambda_1 = 3(\gamma_{\mathrm{ph}} - 1) < 0$, $\lambda_2 = 3(\gamma_{\mathrm{ph}} - \gamma_{\mathrm{DE}}) < 0$, $\lambda_3 = 3\gamma_{\mathrm{ph}} - 4 < 0$, for all admissible $\gamma_{\mathrm{ph}} < 0$, $\gamma_{\mathrm{DE}} > 0$.
$\mathbf{P}_{\mathrm{ph}}$ is a **stable node** — a future attractor (Garcia-Salcedo et al. 2013).

Phase portrait description

This phase diagram (Fig. 1) illustrates the dynamical evolution of the Universe in the *y-z* plane, where $y = \Omega_{DE}$ (the density parameter of standard dark energy) and $z = \Omega_{ph}$ (the density parameter of phantom energy), under the constraint $x + y + z = 1$ (with $x = \Omega_m$ fixed or negligible).
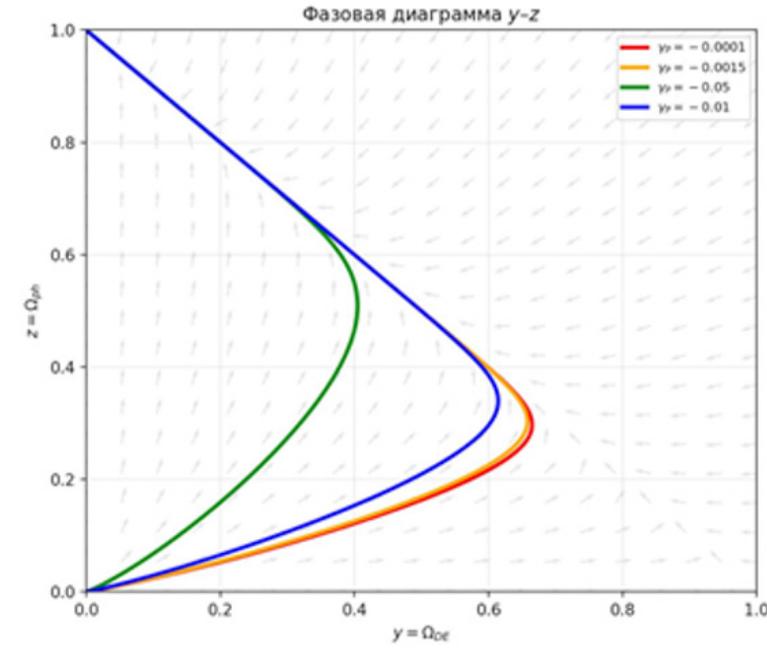

Fig. 1. Phase portrait

The trajectories (colored curves) represent cosmic evolution for different values of the phantom equation-of-state parameter $\gamma_{ph} = 1 + w_{ph}$. All trajectories originate near the origin ($y = 0$, $z = 0$) — corresponding to matter/radiation domination — and converge toward the point ($y = 0$, $z = 1$), which is the **phantom attractor** (Garcia-Salcedo et al. 2013).

The vector field (gray arrows) confirms that the phantom-dominated state is globally stable: all flows are directed toward the upper-left corner. The shape of the trajectories shows that as γ_{ph} becomes more negative (i. e., phantom energy becomes 'stronger'), the transition from standard dark energy to phantom dominance occurs earlier and more rapidly.

This visualization supports the key result: **phantom energy acts as a future attractor**, regardless of its initial abundance or the specific value of $w_{ph} < -1$ (Ashmita et al. 2024; Cai, Saridakis 2009).

Conclusions

The results indicate the following sequence of cosmological epochs:
1. evolution begins near P_r (radiation domination),
2. passes through the saddle point P_m (matter era, structure formation) (Nunes, Mimoso 2000),
3. then through P_{DE} (standard dark energy era) (Copeland et al. 2006),
4. and inevitably approaches P_{ph} (phantom energy domination) (Garcia-Salcedo et al. 2013).

Phantom energy domination ($w_{ph} < -1$) implies monotonic growth of its density, with $\rho_{ph} \propto a^{-3(1 + w_{ph})}$ (where the exponent is positive), leading to a divergence of the scale factor, density, and curvature in finite cosmic time — the 'Big Rip' scenario (Copeland et al. 2006).

Thus, within the considered non-interacting four-component model, phantom energy is a globally stable future attractor. This is rigorously proven: all eigenvalues of the Jacobian at P_{ph} are negative for physically admissible $\gamma_{ph} < 0$ (Garcia-Salcedo et al. 2013). Although the model is primarily pedagogical (due to the absence of interactions and potential tensions with H_0 and S_8 measurements), it demonstrates the fundamental possibility of realising the phantom phase as the natural endpoint of cosmological evolution (Ashmita et al. 2024; Nojiri et al. 2017).

Conflict of Interest

The authors declare that there is no conflict of interest, either existing or potential.

Author Contributions

Vitaly D. Vertogradov supervised the project and provided guidance at all its stages; formulated the physical problem, developed the stability analysis methodology, interpreted the cosmological implications, and revised the manuscript.

Ulyana V. Yamaltdinova performed analytical and numerical calculations, constructed the phase-space portrait, and prepared figures and diagrams.

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