Linear properties of chaotic states of systems described by equations of nonlinear dynamics: Analogy with quantum theory

A. V. Liaptsev[✉1]

[1] Herzen State Pedagogical University of Russia, 48 Moika Emb., Saint Petersburg 191186, Russia

Author

Alexander V. Liaptsev, ORCID: 0000-0002-8702-9062, e-mail: Lav@herzen.spb.ru

Abstract. The paper provides a theoretical exploration of properties of systems described by equations of nonlinear dynamics in a chaotic state. Using the example of a system described by Duffing equations, it is shown that when the state of the system corresponds to a chaotic (strange) attractor, it is possible to determine a function whose meaning corresponds to the probability density. In this case, the resulting equation for the probability density is linear, so that the solution methods developed for linear differential equations, in particular the method of perturbation theory, can be applied to solve the equation in question. This results in a linear dependence of the average values of physical quantities on the parameter that characterizes small perturbations of the system. The numerical experiment confirms this linear relationship.

Keywords: nonlinear dynamics, strange attractor, probability density, chaos, perturbation theory.

Introduction

The specific properties of systems described by nonlinear equations have been known for a long time. One of the first works performed in the 19th century is the observation of solitons—solitary waves on the surface of water, as well as the study of the corresponding solutions of the Korteweg—de Vries equation. Currently, there are independent fields of study that focus on nonlinear waves and signals, such as nonlinear optics and nonlinear dynamics. Today, these fields of study develop largely due to the capabilities of modern computer technology, since it is the combination of analytical and computational research methods that allows us to advance in elucidating the qualitative features inherent in these phenomena. An overview of recent developments in the field of nonlinear dynamics can be found in (Gonchenko et al. 2017; Loskutov 2007). It should also be noted that the availability of sufficiently powerful computers and appropriate software allows us to set tasks related to non-linear systems as educational tasks at universities and even at school.

A characteristic feature of the system described by nonlinear equations is the occurrence of chaos at certain parameters of the system. Despite the determinism of solutions of the equations of classical dynamics, the nonlinearity of these equations leads to the fact that the state of the dynamical system is unpredictable after a characteristic time that determines the horizon of predictability. As noted by G. M. Zaslavsky and R. Z. Sagdeev in one of the first monographs on this topic (Sagdeev et al. 1988), since the trajectories of particles in phase space become in this case extremely complex and confusing, it is useless to monitor each trajectory separately. Instead, one should consider a set of trajectories that at any time occupy a finite volume of phase space, and the distribution of particles in it is characterized by some density.

Calculation of the probability density is a standard problem in statistical physics and quantum mechanics. It should be noted that the equations of quantum theory are linear. This leads to the fact that the evolution of a quantum mechanical system is in a certain sense deterministic. Naturally, this determinism is different from what occurs in classical dynamics. Solutions of the equations do not make it possible to predict at what point in the phase space the system is located at some point in time. However, solving the equations makes it possible to determine the probability that the system under consideration is located in a given region of the phase space at a given time.

The question arises whether it is possible for systems described by classical dynamical equations whose solutions correspond to a chaotic state to formulate equations that determine the probability density, similar to the equations in quantum theory and statistical physics. It turns out that this can be done, at least for the simplest systems. Similar equations were obtained for a rotator in an external harmonic field (Liapzev 2019). Numerical experiments show that solving equations for the probability density gives similar results to those obtained when solving dynamic equations. In this case, the equations for the probability density are linear, which is indicative of determinism similar to that given by the equations of quantum theory.

Linear properties of the equations of quantum theory cause other properties that are convenient for solving equations. In particular, at the beginning of the development of quantum theory, it was possible to formulate a perturbation theory that allows to effectively solve problems that cannot be solved analytically and to obtain qualitative consequences. If a small parameter can be selected in the problem, then the solution of the equations (wave function, probability density, average values of physical quantities) can be obtained as a power expansion of this parameter. However, if the equations for the probability density of classical nonlinear systems in a state of chaos turn out to be linear, then in the presence of a small parameter, perturbation theory should also work, so that small perturbations should lead to small corrections. The average values of physical quantities should be represented as a small parameter expansion. The paper provides an example of the simplest of such systems and tests its properties by a numerical experiment.

The Duffing equation. Regular and chaotic solutions

The Duffing equations describe forced oscillations in a system with a *W*-potential. Note that the system is highly visual, so it may find use in the educational process (Kondrat'ev, Lyaptsev 2008). In reality, it corresponds to a load that is located in the upper part of the elastic rod and swings in the same plane due to an external force (see Fig. 1). In the absence of external influence, the system has two stable equilibrium positions (*b* and *d*) and one unstable equilibrium position (*c*).
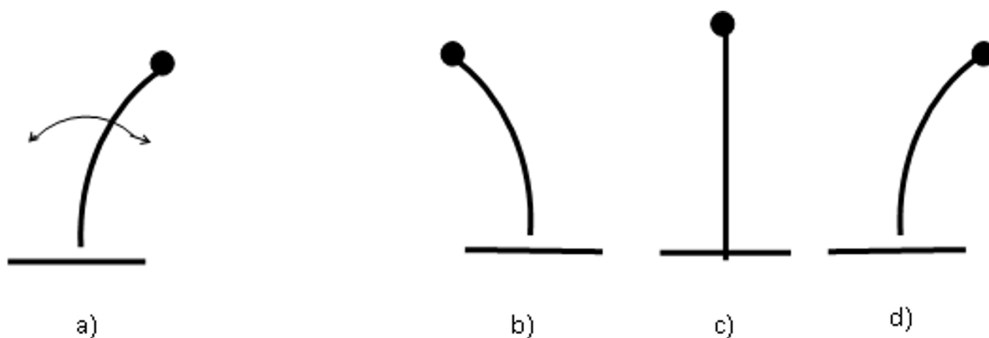

Fig. 1. Model corresponding to the Duffing equation

The oscillation equations have the form:

$$\ddot{x}(t) + \gamma \dot{x}(t) - \alpha x(t) + \beta x^3(t) = F \sin(\omega t). \tag{1}$$

In this equation, the constants α and β characterize the nonlinear elastic force, the constants F and ω—the amplitude and frequency of the external force, and the constant γ—the energy dissipation due to the non-conservative viscous friction force.

Equation (1) can be reduced to a system of three autonomous differential equations of the 1st order:

$$\dot{x} = \upsilon,$$
$$\dot{\upsilon} = F \sin \varphi + \alpha x - \beta x^3 - \gamma \upsilon, \qquad (2)$$
$$\dot{\varphi} = \omega.$$

Since the system under consideration is dissipative, for any initial conditions, the region of phase space in which the state of the system is located narrows, over time, to the attractor. In the case of regular periodic solutions, the attractor is a closed loop in the phase space. However, for some parameter values, the solution tends to a chaotic (strange) attractor. The dimension of the strange attractor is fractional, and a set of points that is a fractal appears in the section of the phase space with a certain plane. A characteristic property of this system of equations is that with a smooth (adiabatic) change in the control parameter *F*, the state of the system jumps from chaotic motion to regular periodic motion.

The system of equations (2) has a certain symmetry. This means that the system of equations remains invariant with respect to some transformation of variables. In addition to the obvious symmetry $\phi \rightarrow \phi + 2\pi$, there is a symmetry with respect to the set of transformations:

$$x \rightarrow -x, \quad \upsilon \rightarrow -\upsilon, \quad \varphi \rightarrow \varphi + \pi. \qquad (3)$$

Together with the identity transformation, this set of transformations forms a symmetry group isomorphic to the point group $\boldsymbol{C}_s$. Note that a similar system can be considered in the equations of quantum mechanics. In this case, the non-stationary Schrodinger equation determines the eigenvalues of quasi-energies, and the corresponding quasi-energy states are transformed by irreducible representations of the above group, that is, they are even or odd. In this classical case, equations (2) do not imply any properties related to symmetry due to the nonlinearity of the equations. However, numerical calculations (Liaptsev 2013; 2014a; 2014b; 2015) show that the Poincare sections of the phase space with parameters corresponding to the chaotic state are fully symmetric with respect to the transformations (3).

Probabilistic approach

Let us now define a density probability distribution $\rho(\phi, \upsilon, x)$ as follows: for a given value ϕ the value $\Delta w = \rho(\phi, x, \upsilon)\Delta x \Delta \upsilon$ is equal to the probability that the trajectory of the system passes in the region $[x, x + \Delta x; \upsilon, \upsilon + \Delta \upsilon]$ and the normalization condition is set:

$$\int dx \int d\upsilon \rho (\varphi, x, \upsilon) = 1, \quad \forall \varphi.$$

Note that the density of the probability distribution can be approximately obtained by a numerical experiment. For this purpose, the section plane needs to be broken into cells , then we perform calculations $x(t)$ and $v(t)$ over large enough time and put ρ for each cell which is proportional to the number of points that fell into the cell.

The equation for the probability density can be obtained from the Hamilton equations for a classical mechanical system. The derivation of such an equation for a rotator in an external harmonic field is described in detail in (Liapzev 2019). Omitting the derivation of the equation for the system considered here, we give the final expression:

$$\omega \frac{\partial \rho}{\partial \varphi} + \upsilon \frac{\partial \rho}{\partial x} + \left(F\sin\varphi + \alpha x - \beta x^3 - \gamma \upsilon \right) \frac{\partial \rho}{\partial \upsilon} = 0. \qquad (4)$$

Equation (4) is a linear first-order partial differential equation. From the theory of differential equations (see (Kamke 1967)) it follows that there are systems of ordinary differential equations associated with a similar differential equation, which, in general, can be written as:

$$\sum_{v=1}^{n} f_v(x_1, x_2, ..., x_n) \frac{\partial \psi(x_1, x_2, ..., x_n)}{\partial x_v} = 0. \qquad (5)$$

The ordinary differential equations have the form:

$$\dot{x}_\nu(t) = f_\nu(x_1, x_2, ..., x_n), \quad \nu = 1, 2, ..., n. \tag{6}$$

Characteristic curves defined by the system of equations (6) are associated with solutions of equation (5). In particular, if $\phi_\nu(t)$ is the solution of a system of equations (6), then if, and only if, the function $\psi(x_1, x_2, ..., x_n)$ is the integral of equation (5) when $\psi(\varphi_1(t), \varphi_2(t), ..., \varphi_n(t)) = \text{const}$ for any characteristic curve. It is easy to verify that in the case under consideration the system of equations that determines the characteristic curves of the equation (4) exactly coincides with the system of equations (2) that determines the dynamics of the system of interest.

One of the methods for solving equations of the form (5) is associated with finding characteristic curves. In our case it can be implemented as finding functions $\rho(\phi, \upsilon, x)$ in the above numerical experiment. However, this is not the only solution. The linear properties of equation (4) allow us to use the Fourier transform for its solution. Since all variables included in the equation are restricted to certain values, the Fourier transform reduces to a multidimensional (in this case, three-dimensional) Fourier series. The numerical solution of the equation can be obtained by restricting the series with some finite values, resulting in a system of linear equations. The accuracy of solutions increases with increasing values that limit the series. In paper (Liapzev 2019), it is shown that the solution obtained by the Fourier transform method for the rotator equation gives a picture of the Poincare sections of the corresponding chaotic attractor similar to the picture obtained by solving dynamic equations.

Perturbation theory for the probability density equation

As mentioned above, the linearity of the equations allows us to obtain their solutions and expressions for the average values of some operators in the form of power expansions of a small parameter. Let us denote the differential operator included in equation (4) by L:

$$L = \omega \frac{\partial}{\partial \varphi} + \upsilon \frac{\partial}{\partial x} + \left(F \sin \varphi + \alpha x - \beta x^3 - \gamma \upsilon\right) \frac{\partial}{\partial \rho},$$

then the equation takes a simple form:

$$L\rho(\varphi, x, \upsilon) = 0.$$

We now add a small perturbation of the form λV, where V is a certain operator that depends on the variables ϕ, x, υ and the corresponding partial derivatives, and λ is a small parameter introduced to obtain expansions in its powers. Find the function $\rho(\phi, \upsilon, x)$ as a power expansion of this parameter:

$$\rho(\varphi, x, \upsilon) = \sum_{n=0}^{N} \lambda^n \rho^{(n)}(\varphi, x, \upsilon). \tag{7}$$

The maximum value of degree of parameter λ in expansion (7) is determined not so much by the accuracy with which the calculation is performed, but, rather, by the qualitative effects obtained at certain degrees of the parameter. For example, in the theory of molecular spectra, the effective small parameter included in the quantum molecular Hamiltonian is the Born—Oppenheimer parameter: $\kappa = \sqrt[4]{m / M}$, where m is the mass of the electron and M is the average mass of the molecule nuclei. In the zero approximation for this parameter, the electronic, vibrational, translational, and rotational motions of the molecule are separated, and the effects due to the interaction of these motions are manifested in certain degrees of expansion of eigenvalues and eigenfunctions on this parameter (Kiselev, Lyapcev 1989).

Finding the terms of the expansion (7) is reduced to a sequential separation from the equation:

$$(L + \lambda V) \sum_{n=0}^{N} \rho^{(n)}(\varphi, x, \upsilon) = 0, \tag{8}$$

equations for a given degree of the parameter λ. In zero order, we get the equation:

$$L\rho^{(0)}(\varphi, x, \upsilon) = 0,$$

corresponding to equation (4). The equation of the 1ˢᵗ order:

$$L\rho^{(1)}(\varphi,x,\upsilon)+V\rho^{(0)}(\varphi,x,\upsilon)=0$$

allows us to find the first term of the probability density expansion:

$$\rho^{(1)}=-L^{-1}V\rho^{(0)},$$

where L^{-1} on the right side of the equality is the inverse operator of L. Similarly, the following terms of the expansion of the function $\rho(\phi, \upsilon, x)$ can be found.

The solution of the problem by the perturbation method is thus reduced to finding some inverse operators that appear in equations of a given order. In quantum theory, such operators are found as a result of expansion of functions on the eigenfunctions of an unperturbed operator. Here, other methods can be used, for example the Fourier transform mentioned above. It is important that the average value of any operator can also be obtained as a power expansion of a small parameter:

$$<A>=\sum_{n=0}^{N}\lambda^{n}A^{(n)},$$
$$A^{(n)}=\int A(\varphi,x,\upsilon)\rho^{(n)}(\varphi,x,\upsilon)\,d\varphi dx d\upsilon.$$

(9)

Despite the fact that formally the decomposition of the average values of operators begins with zero order over a small parameter, the symmetry properties of the systems under consideration can lead to zero values of the first few terms of the decompositions of some operators. In particular, the zero term of the expansion turns out to be zero if the corresponding operator is not transformed by the full-symmetric representation of the symmetry group of the system under consideration.

Turning to the problem we are considering, it is easy to notice that one of the operators that is transformed by the odd representation of the C_s group is the operator of coordinate x (see transformations (3)). It follows that the average value of the x coordinate for the chaotic state of the system described by the Duffing equation turns out to be zero. A numerical experiment confirms this conclusion. Calculating the average value for a finite time interval gives an average of zero, although due to the statistical properties of the state under consideration, each value differs slightly from zero, that is, there are fluctuations in the zero-average value. A non-zero mean value of the coordinate should be obtained if a perturbation is added to the unperturbed operator L, which disturbs the symmetry of the system. One can offer several similar operators. One of these operators is the operator corresponding to adding an odd degree to the W-potential. This corresponds to adding an even degree to the corresponding potential force. Accordingly, the perturbation operator included in equation (4) will have the form:

$$V_1=x^2\frac{\partial}{\partial\upsilon}\cdot$$

(10)

It is easy to verify that the operator changes its sign during transformations (3). Another possible way to break the symmetry is to add a component with doubled frequency to the external driving force:

Accordingly, the perturbation operator included in equation (4) will have the form:

$$V_2=\sin(2\varphi)\frac{\partial}{\partial\upsilon}\cdot$$

(11)

Note that a linear combination of these operators can be used as a perturbation. Then the first order corrections to the average value of the x coordinate will be expressed as a linear function of the corresponding small parameters. In the second order of the perturbation theory, corrections that are a quadratic function of small parameters will appear, and so on.

We denote small parameters for operators V_1 and V_2 by λ_1 and λ_2. Then the system of equations (2) takes the form:

$$\dot{x}=\upsilon$$
$$\dot{\upsilon}=F\sin\varphi+\alpha x-\beta x^3-\gamma\upsilon+\lambda_1 x^2+\lambda_2\sin(2\varphi),$$
$$\dot{\varphi}=\omega$$

(12)

and the equation for the probability density is:

$$\omega \frac{\partial \rho}{\partial \varphi} + \upsilon \frac{\partial \rho}{\partial x} + \left(F\sin\varphi + \alpha x - \beta x^3 - \gamma\upsilon + \lambda_1 x^2 + \lambda_2 \sin(2\varphi) \right)\frac{\partial \rho}{\partial \upsilon} = 0 \ . \tag{13}$$

The average values of the x coordinate can be calculated either by applying perturbation theory to equation (13) and then using averaging by formula (9), or by solving the system of equations (12) and then averaging the function $x(t)$ over time. Note that it does not follow from the system (12) itself that in the case of a chaotic solution of the system, averaging the coordinate over time for different values of λ_1 and λ_2 will give a linear function of these parameters. Thus, a numerical experiment consisting in obtaining the average coordinate value by solving the system (12) is actually equivalent to a real experiment in relation to proving the applicability of the equation for probability density to the system described by the Duffing equation.

Results of a numerical experiment

We present the results of a numerical experiment for the next values of the parameters included in the system of equations (12): $\alpha = \beta = 1$, $\omega = 1.2$, $\gamma = 0.1$. The value of the parameter F was chosen in such a way that the solution of the system of equations remained chaotic in a certain range of parameters λ_1 and λ_2. Collapse to a regular (periodic) solution for certain values of parameters led to the fact that the corresponding solution was unsuitable for conducting a numerical experiment, since the equation for the probability density is not applicable if the attractor of the equations is a closed line. Below we present the results of a numerical experiment for the values: $F = 5.4$ and $F = 8.7$. In both cases, the averaging was performed over a time interval equal to 10.000 periods of external force, respectively: $\Delta t = 5.236 \cdot 10^4$.

The results of the numerical experiment can be represented as:

$$<x> = y_1\lambda_1 + y_2\lambda_2 ,$$

where the coefficients y_1 and y_2 were determined from the calculated values for a certain set of parameters λ_1 and λ_2 by the method of least squares. The least squares method also makes it possible to identify the error in determining these coefficients.

Results for F = 5.4.

Parameter λ_1 changed from -0.4 to 0.4 in increments of 0.1, parameter λ_2 changed from -0.15 to 0.15 in increments of 0.05. The results are shown in Table 1.

The deviations of the calculated values from the plane defined by the parameters y_1 and y_2 can be clearly seen in Fig. 2, where the calculated values are shown in circles.

Table 1. Calculation results for F = 5.4

	Values of coefficients	Absolute error	Relative error
Calculation for $\lambda_2 = 0$ and variation of λ_1	$y_1 = 0.344$	$\Delta y_1 = 0.005$	$\delta y_1 = 0.01$
Calculation for $\lambda_1 = 0$ and variation of λ_2	$y_2 = -0.36$	$\Delta y_2 = 0.02$	$\delta y_2 = 0.04$
Calculation for variation of λ_1 and λ_2	$y_1 = 0.301$ $y_2 = -0.37$	$\Delta y_1 = 0.004$ $\Delta y_2 = 0.01$	$\delta y_1 = 0.016$ $\delta y_2 = 0.03$

Results for F = 8.7.

Parameter λ_1 changed from -0.4 to 0.4 in increments of 0.1, parameter λ_2 changed from -0.6 to 0.6 in increments of 0.2. The results are shown in Table 2.

Table 2. Calculation results for F = 8.7

	Values of coefficients	Absolute error	Relative error
Calculation for $\lambda_2 = 0$ and variation of λ_1	$y_1 = 0.70$	$\Delta y_1 = 0.02$	$\delta y_1 = 0.03$
Calculation for $\lambda_1 = 0$ and variation of λ_2	$y_2 = -0.25$	$\Delta y_2 = 0.01$	$\delta y_2 = 0.04$
Calculation for variation of λ_1 and λ_2	$y_1 = 0.76$ $y_2 = -0.26$	$\Delta y_1 = 0.01$ $\Delta y_2 = 0.01$	$\delta y_1 = 0.02$ $\delta y_2 = 0.03$

The deviations of the calculated values from the plane defined by the parameters y_1 and y_2 can be clearly seen in Fig. 3, where the calculated values are shown in circles.

As can be seen from the above results in a good approximation, with an accuracy of several percent, a linear dependence of the average values of the x coordinate on the parameters λ_1 and λ_2 is observed.
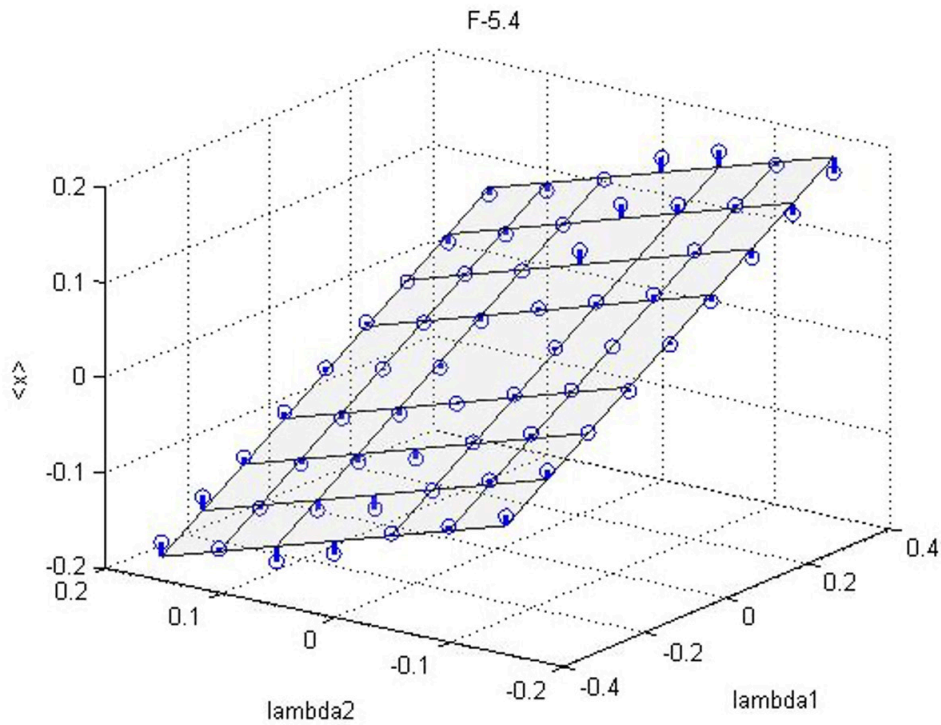

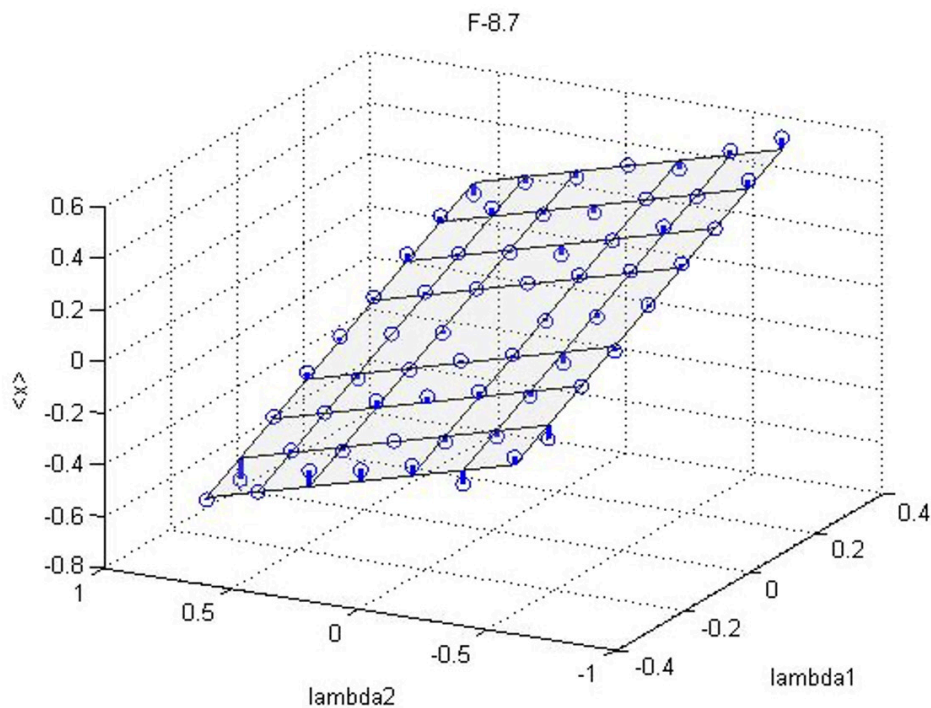Fig. 2. The deviation of numerical results from the linear dependence at F = 5.4

Fig. 3. The deviation of numerical results from the linear dependence at F = 8.7

Conclusion

The results of the numerical experiment show that small perturbations in the system described by the Duffing equation lead to a linear dependence of the average values of physical quantities on small parameters that characterize the perturbation in the chaotic behavior of solutions. This indicates the applicability of perturbation theory to such systems for chaotic solutions. This fact, in turn, can be explained by the fact that the chaotic state of the system can be characterized by a probability density that satisfies the linear equation. Due to the linearity of this equation for small perturbations, the probability density, and, consequently, the average values of physical quantities can be represented as a decomposition in small parameters that characterize the perturbations.

The fact that the chaotic state of a system described by nonlinear dynamic equations can be characterized by a probability density indicates that, despite the non-determinism of classical solutions, in the sense of probability, the chaotic regime is determined to the same extent as the state of the system described in the language of quantum theory.

References

Gonchenko, A. S., Gonchenko, S. V., Kazakov, A. O., Kozlov, A. D. (2017) Matematicheskaya teoriya dinamicheskogo khaosa i ee prilozheniya: Obzor. Chast' 1. Psevdogiperbolicheskie attraktory [Mathematical theory of dynamical chaos and its applications: Review. Part 1. Pseudohyperbolic attractors. Pseudohyperbolic attractors]. *Izvestiya Vysshikh uchebnykh zavedeniy. Prikladnaya nelineynaya dinamika — Izvestiya VUZ. Applied Nonlinear Dynamics*, 25 (2), 4–36. (In Russian)

Kamke, E. (1971) *Differentialgleichungen: Lösungsmethoden und Lösungen. Vol. 2.* Leipzig: Akademische Verlagsgesellschaft, 243 p. (In German)

Kiselev, A. A., Lyapcev, A. V. (1989) *Kvantovomekhanicheskaya teoriya vozmushchenij. Diagrammnyj metod [Quantum mechanical perturbation theory. Diagram method].* Leningrad: Leningrad State University Publ., 360 p. (In Russian)

Kondrat'ev, A. S., Lyaptsev, A. V. (2008) *Fizika. Zadachi na komp'yutere [Physics. Tasks on the computer].* Moscow: Fizmatlit Publ., 400 p. (In Russian)

Liaptsev, A. V. (2013) Simmetriya regulyarnykh i khaoticheskikh dvizhenij v zadachakh nelinejnoj dinamiki. Uravnenie Duffinga [The symmetry of regular and chaotic motions in nonlinear dynamics problems. Duffing Equation]. *Izvestia Rossijskogo gosudarstvennogo pedagogicheskogo universiteta im. A. I. Gertsena — Izvestia: Herzen University Journal of Humanities & Sciences*, 157, 24–34. (In Russian)

Liaptsev, A. V. (2014a) Simmetriya regulyarnykh i khaoticheskikh dvizhenij v zadachakh nelinejnoj dinamiki. Rotator v periodicheskom pole [Symmetry of regular and chaotic motions in nonlinear dynamic problems. Rotator in periodic field]. *Izvestia Rossijskogo gosudarstvennogo pedagogicheskogo universiteta im. A. I. Gertsena — Izvestia: Herzen University Journal of Humanities & Sciences*, 165, 23–35. (In Russian)

Liaptsev, A. V. (2014b) Simmetriya v zadachakh nelinejnoj dinamiki. Proyavlenie svojstv simmetrii v polyarizatsii izlucheniya [Symmetry in problems of nonlinear dynamics. The manifestation of the properties of the symmetry in the polarization of radiation]. *Izvestia Rossijskogo gosudarstvennogo pedagogicheskogo universiteta im. A. I. Gertsena — Izvestia: Herzen University Journal of Humanities & Sciences*, 168, 16–28. (In Russian)

Liaptsev, A. V. (2015) Proyavlenie svojstv simmetrii v zadachakh nelinejnoj dinamiki. Effekty, analogichnye vyrozhdeniyu v kvantovomekhanicheskikh zadachakh [Manifestation of symmetry properties in problems of nonlinear dynamics. Effects analogous to degeneration in quantum mecanical problems]. *Izvestia Rossijskogo gosudarstvennogo pedagogicheskogo universiteta im. A. I. Gertsena — Izvestia: Herzen University Journal of Humanities & Sciences*, 173, 64–77. (In Russian)

Liapzev, A. V. (2019) The calculation of the probability density in phase space of a chaotic system on the example of rotator in the harmonic field. *Computer Assisted Mathematics*, 1, 55–65. (In English)

Loskutov, A. Yu. (2007) Dynamical chaos: Systems of classical mechanics. *Physics-Uspekhi*, 50 (9), 939–964. DOI: 10.1070/PU2007v050n09ABEH006341 (In English)

Sagdeev, R. Z., Usikov, D. A, Zaslavskii, G. M. (1988) *Nonlinear physics: From the pendulum to turbulence and chaos.* 2nd ed. Chur: Harwood Academic Publ., 675 p. (In English)