Physics of Complex Systems, 2021, vol. 2, no. 1
www.physcomsys.ru

Check for updates

Theoretical Physics. Cosmology

UDC 524.8 https://www.doi.org/10.33910/2687-153X-2021-2-1-33-40

The nature of the naked singularity in generalized Vaidya spacetime and white hole geodesics

V. D. Vertogradov[✉1, 2]

[1] Herzen State Pedagogical University of Russia, 48 Moika Emb., Saint Petersburg 191186, Russia
[2] Special Astrophysical Observatory of the Russian Academy of Sciences, St Petersburg branch, 65 Pulkovskoe Rd, Saint Petersburg 196140, Russia

Author
Vitalii D. Vertogradov, ORCID: 0000-0002-5096-7696, e-mail: vdvertogradov@gmail.com

For citation: Vertogradov, V. D. (2021) The nature of the naked singularity in generalized Vaidya spacetime and white hole geodesics. *Physics of Complex Systems*, 2 (1), 33–40. https://www.doi.org/10.33910/2687-153X-2021-2-1-33-40

Received 4 December 2020; reviewed 30 December 2020; accepted 6 January 2021.

Funding: This work was supported by RFBR grant No. 18-02-00461.

Abstract. This paper gives conditions to the energy-momentum tensor when the gravitational collapse of generalized Vaidya spacetime results in a naked singularity. We also consider the gravitational collapse of a matter cloud whose interior is described with Vaydia-de Sitter spacetime—when the collapse ends, this metric must be the de Sitter one, and the result of such a collapse is a so-called regular black hole. We provide the geodesic equations for the case when the matter is described with the equation of the state both $P = -\rho$ and $P = \alpha\rho$, where α belongs to the interval $\left(0, \frac{1}{3}\right]$. We match these geodesics with geodesics in the Schwarzschild metric. We also explain the nature of white hole geodesics with either the naked singularity formation or the regular black hole formation.

Keywords: generalized Vaidya spacetime, naked singularity, geodesics, matching, white hole geodesics, black hole.

Introduction

The problem of gravitational collapse is one of the most interesting in modern theoretical physics. In 1939, Oppenheimer and Snyder (Oppenheimer, Snyder 1939) built a model of the gravitational collapse with the assumption of the pressureless matter. For a long time, a black hole was considered to be the inevitable result of the gravitational collapse. However, it was shown that the gravitational collapse might also result in a naked singularity (see (Joshi 2007; Joshi, Malafarina 2011)). The naked singularity forms when the time of the singularity formation during the gravitational collapse is less than the time of the apparent horizon formation, and there must be a family of non-spacelike, future-directed geodesics which terminate at the central singularity in the past. It should be noted that the existence of only a family of non-spacelike geodesics is not sufficient for a naked singularity formation. It was shown (Grib et al. 2014; Vertogradov 2015) that all geodesics for particles with negative energy in the ergosphere of a rotating black hole appear in the ergosphere from the region inside the gravitational radius. In particular, it was shown that all null geodesics for such particles originate at the singularity of the Kerr black hole. And this statement is valid for geodesics for particles with positive energy (but not for all) (Grib, Pavlov 2015). However, in the case when $a \neq 1$ (i.e., a non-critical black hole), the Kerr black hole is not a naked singularity. If we can prove only the existence of a family of non-spacelike geodesics, then the result of a gravitational collapse might be a white hole. But, in the case of white hole, it takes infinite time to get to the external observer, because a white hole has the apparent horizon. It is worth mentioning that a naked singularity formation violates cosmic censorship conjecture (CCC) which states that all singularity must be covered with a horizon. However, Papapetrou (Papapetrou 1985) was the first who showed the violation of CCC, and the Vaidya spacetime was one of the earliest counter examples of CCC. The generalized Vaidya spacetimes have been widely used in the studies of dynamical black holes (Dawood, Ghosh 2004)), black holes with trapped regions, and the gravitational collapse (Brassel, Goswami, Maharaj 2017; Brassel, Maharaj, Goswami 2017).

Recently, Maharaj (Mkenyeleye et al. 2014) has shown that the result of the gravitational collapse of generalized Vaidya spacetime might be the naked singularity and the result of this collapse depends on the mass function $M(v,r)$, where v is advanced Eddington time. The conditions on the mass function, when the result of such a collapse is a naked singularity, were given in our earlier paper (Vertogradov 2016). Further, it was shown that this kind of singularity might be gravitationally strong (Nolan 1999; Tipler 1977). An interesting question arises: what matter is required in order for a nacked singularity to be formed? To answer this question, the expression of the energy-momentum tensor was investigated, and it was shown that a naked singularity might be formed when type II of matter field (Hawking, Ellis 1973) dominates in the center of the collapsing matter cloud.

There is also another interesting question regarding the nature of white hole geodesics. According to geodesics completeness, there must be geodesics which appear in our universe from a region which is located inside the gravitational radius. For example, these geodesics can be the ones for particles with negative energy in the Kerr metric (Grib et al. 2014; Vertogradov 2015), or the ones which describe the motion of an object from the point of rest to the region inside the event horizon. In order to investigate this question, we should consider the past of such geodesics. But black holes are not eternal and, to extend our geodesics into the past, we should consider the gravitational collapse. The nature of white hole geodesics might be either a naked singularity formation during the gravitational collapse or regular black holes (Dymnikova 1992; Dymnikova 2002). In the latter case, the geodesics do not terminate at the singularity in the past, and they come from the infinity but, due to high negative density, turn back to our world. In this aim, we consider the gravitational collapse of the matter cloud whose interior is described with Vaydia—de Sitter spacetime, and whose exterior, with the Schwarzschild one. Also we write down geodesics in this metric and geodesics with the mass function $M(v,r)=C(v)+D(v)r^{1-2\alpha}$ (Vertogradov 2016). Unfortunately, when we have to deal with generalized Vaidya spacetime, we must consider a thin matter layer between two solutions—the generalized Vaidya spacetime and the Schwarzschild one. This is because the generalized Vaidya spacetime in which the mass function M does not depend on the time v is the Schwarzschild spacetime plus some extra term which we can define from the equation of the state. So we cannot smoothly match the generalized Vaidya spacetime with the Schwarzschild metric. In order to explain the nature of a so-called white hole geodesic in the Schwarzschild metric, we will use the usual Vaidya metric when the mass function $M = M(v)$ is the function of time only.

In sec. "Energy momentum tensor in the case of a naked singularity formation" we consider the energy-momentum tensor in the case of the naked singularity formation. In sec. "Vaydia—de Sitter spacetime", we consider the Vaydia—de Sitter metric. In sec. "Matching", we write down geodesics in the generalized Vaidya spacetime and the usual Vaiday spacetime and match them to the Schwarzschild ones.

The system of units $c = G = 1$ will be used throughout the paper. Dash and overdot denote partial derivative with respect to coordinates r and v respectively.

Energy momentum tensor in the case of a naked singularity formation

In the general case, the generalized Vaidya spacetime is given by:

$$
\begin{aligned}
&ds^2 = -\left(1-\frac{2M(v,r)}{r}\right)dv^2 + 2\epsilon dv dr + r^2 d\Omega^2 , \\
&d\Omega^2 = d\theta^2 + \sin^2\theta d\varphi^2 , \\
&\epsilon = \pm 1 ,
\end{aligned} \tag{1}
$$

where $M(v,r)$ is the mass function.

In this paper we consider the energy-momentum tensor which is the combination of types I and II of matter fields (Hawking, Ellis 1973). Here type II of the matter field represents so-called null dust with the energy density μ, and type I is the cosmic strings. In the case of metric (1), by virtue of the Einstein field equations, the energy-momentum tensor has the form (Wang, Wu 1999):

 https://www.doi.org/10.33910/2687-153X-2021-2-1-33-40

$$T_{\mu\nu} = T_{\mu\nu}^{(m)} + T_{\mu\nu}^{(n)} \ , \tag{2}$$

where $T_{\mu\nu}^{(m)}$ and $T_{\mu\nu}^{(n)}$ correspond to types I and II of matter fields respectively.
Now we write down the explicit form of the energy-momentum tensor:

$$\begin{aligned}
&T_{\mu\nu}^{(n)} = \mu L_\mu L_\nu \ , \\
&T_{\mu\nu}^{(m)} = (\rho + P)(L_\mu N_\nu + L_\nu N_\mu) + g_{\mu\nu} P \ , \\
&\mu = \frac{2\dot{M}}{r^2} \ , \\
&\rho = \frac{2M'}{r^2} \ , \\
&P = -\frac{M''}{r} \ , \\
&L_\mu = \delta_\mu^0 \ , \\
&N_\mu = \frac{1}{2}\left(1 - \frac{2M}{r}\right)\delta_\mu^0 - \varepsilon \delta_\mu^1 \ , \\
&L_\mu L^\mu = N_\mu N^\mu = 0 \ , \\
&L_\mu N^\mu = -1 \ .
\end{aligned} \tag{3}$$

Here P is the pressure, ρ is the density and L^μ, n^μ are two null vectors. One should also distinguish the null dust energy density μ, which is the function, and subscript μ which takes values {0, 1, 2, 3}.

Our earlier paper (Vertogradov 2016) gives conditions for the mass function when the result of collapse is a naked singularity. The matter must satisfy the equation of the state $P = \rho\alpha$, where α belongs to the interval $\left(0, \frac{1}{3}\right]$. In this case, the mass function is given by:

$$M(v,r) = C(v) + D(v) r^{1-2\alpha} \ , \tag{4}$$

where C and D are two arbitrary functions of v only. Our model must be physically reasonable, and the energy-momentum tensor must satisfy energy conditions (Poisson 2004). Due to this conditions, we have the following restrictions on the mass function:

$$\begin{aligned}
&C(0) = 0 \ , \ D(0) \geq 0 \ , \ \dot{C}(v) \geq 0 \ , \\
&\dot{C}(v) + \dot{D}(v) r^{1-2\alpha} > 0 \ .
\end{aligned} \tag{5}$$

Now, substituting 4 into 3, we obtain:

$$\begin{aligned}
T_{00} &= \left(\frac{2\dot{C}(v) + \dot{D}(v) r^{1-2\alpha}}{r^2}\right)^{(n)} + \\
&+ \left[\left(1 - \frac{2C(v) + 2D(v) r^{1-2\alpha}}{r}\right)\frac{2(1-2\alpha)D(v)}{r^{1+2\alpha}}\right]^{(m)} \ , \\
T_{01} &= -\rho = -\frac{2(1-2\alpha)D(v)}{r^{1+2\alpha}} \ , \\
P &= \frac{2\alpha(1-2\alpha)D(v)}{r^{1+2\alpha}} \ .
\end{aligned} \tag{6}$$

Here symbols n, m correspond to types I and II of matter fields respectively.

The singularity is formed at $v = 0$, $r = 0$. The necessary condition for a singularity to be naked (Vertogradov 2016) is $D(0) = 0$. Substituting this condition into (6), we find that the part of the energy momentum tensor which corresponds to type I of the matter field vanishes throughout except for the center at $r = 0$ in this case. Thus, the singularity can be naked only when $T^{(m)}_{\mu\nu} = 0$. If $T^{(m)}_{\mu\nu} \neq 0$, the apparent horizon is formed and the singularity stops to be naked.

If we take into account quantum effects, a naked singularity can exist in the present of type I of the apparent horizon, but its formation is still temporary, and in short time the apparent horizon is formed, and the singularity becomes covered with the horizon.

So, in this case, a naked singularity is formed only if type I of the matter field is present at the singularity—then there is the level $0 < r \leq r_{II}$ which purely contains of type II of the matter field (here r_{ii} corresponds to the end of this level). So we must find out the solution of Einstein equations for this level, then we must match this new solution with the generalized Vaidya solution at $r = r_{ii}$.

The Einstein tensor in the case of generalized Vaidya spacetime has components:

$$
\begin{aligned}
G_{00} &= \frac{\left(2M - r\right)M'' + 2\dot{M}}{r^2} + \frac{1}{2}\left(1 - \frac{2M}{r}\right)\left(\frac{2M''}{r} + \frac{4M'}{r^2}\right), \\
G_{01} &= -\frac{2M'}{r^2}, \\
G_{22} &= -2rM'', \\
G_{33} &= \sin^2\theta G_{22} .
\end{aligned}
\tag{7}
$$

From (3) we see that in the case of type II of the matter field we have only one non-vanishing component of energy-momentum tensor—that is, T_{00}. Hence, from the fact that $G_{01}=T_{01}=0$, we obtain:

$$
\begin{aligned}
&2\frac{2M'}{r^2} = 0, \\
&M = M\left(v\right).
\end{aligned}
\tag{8}
$$

So the level $0 < r \leq r_{II}$ is described with the Vaidya metric.

In this case, at the border $r = r_{ii}$ of the two levels we must have:

$$
M\left(v\right) = C\left(v\right) + D\left(v\right) r_{II}^{1-2\alpha} .
\tag{9}
$$

Vaydia—de Sitter spacetime

The expressions for the density and the pressure in the case of generalized Vaidya spacetime have the form:

$$
\begin{aligned}
\rho &= \frac{2m'}{r^2}, \\
P &= -\frac{m''}{r}.
\end{aligned}
\tag{10}
$$

We are interested in the equation of the state:

$$
P = -\rho .
\tag{11}
$$

In this case the mass function has the form:

$$
m\left(v, r\right) = c\left(v\right) + d\left(v\right) r^3 .
\tag{12}
$$

The exterior metric after collapse must be the de Sitter one. The first requirement for this is:

$$
c\left(v\right) = 0 .
\tag{13}
$$

If we cannot satisfy this condition, then the metric will have singularity at $r = 0$, and, as a result, it will not be the de Sitter one.

 https://www.doi.org/10.33910/2687-153X-2021-2-1-33-40

Then, at the time $v = v_{end}$, when the gravitational collapse stops, the following condition must be met:

$$d\left(v_{end}\right) = \frac{\lambda}{6} . \tag{14}$$

Here λ is the cosmological constant.

The equation of the apparent horizon in this case is given by:

$$r = \frac{1}{d\left(v\right)} . \tag{15}$$

The Vaydia—de Sitter spacetime is matched with the Schwarzschild one at the boundary of the collapsing cloud $r = r_b$, and following conditions must be satisfied:

$$\begin{gathered} d\left(v\right) r_b^3 = M = const , \\ dv = dt + \frac{1}{1-\frac{2M}{r}} dr . \end{gathered} \tag{16}$$

As seen from (16), we have a narrow class of functions to match these metrics, because we cannot smoothly match geodesics at the boundary of the two metrics. Because $\frac{d}{dr}\left(D\left(v\right)r^3\right)|_{r=r_b} \neq 0$, as in the case of Schwarzschild spacetime when $\frac{dM}{dr} = 0$, there must be a thin matter layer between these spacetimes which smoothly matches to both metrics.

Let us consider the region $0 < r < r_{VD}$ which is described by Vaydia-de Sitter spacetime. r_{VD} is the boundary of this metric. We know that the metric of this region is:

$$ds_{DV}^2 = -\left(1 - 2\, d\left(v\right) r^2\right) dv^2 + 2dvdr + r^2 d\Omega^2 . \tag{17}$$

Now let us consider a family of radially null geodesics in this metric. We can write:

$$\frac{dv}{dr} = \frac{2}{1-d\left(v\right)r^2} . \tag{18}$$

The apparent horizon in this metric is given by:

$$r_{AH} = \sqrt{\frac{1}{2d\left(v\right)}} . \tag{19}$$

Also we must notice that the expansion is positive inside the apparent horizon and negative, outside it. The expansion θ in this case is written by:

$$e^{-\gamma}\theta = \frac{2}{r^2}\left(1-2d\left(v\right)r^2\right) , \tag{20}$$

where the factor $e^{-\gamma}$ does not any impact on the sign of the expansion.

Now, if we assume that $r_{AH} < r_{VD}$, we can consider a geodesic which originates at the infinity in Schwarzschild spacetime and then crosses the boundary of the collapsing cloud whose interior is described by the Vaydia—de Sitter metric. Now let us look at the equation (18): we will notice that this geodesic cannot cross the apparent horizon because in this case $\frac{dv}{dr}$ must be negative (we require only forward motion in time), but inside the apparent horizon $\frac{dv}{dr}$ is always positive. If $r_{AH} \geq r_{VD}$, then the geodesic has a turning point inside the thin matter layer because it cannot cross the hypersurface $r = r_{VD}$. If such a geodesic then goes to Schwarzschild spacetime, then this geodesic becomes a white hole geodesic in Schwarzschild spacetime.

Matching

We begin our consideration with the simplest case of radial null geodesics. In Schwarzschild spacetime, radial null geodesics are given by:

$$\frac{dt}{dr} = \pm \frac{1}{1 - \frac{2M}{r}} \ . \tag{21}$$

Radial null geodesics in the case of generalized Vaidya spacetime with the mass function $M(v,r) = C(v)+D(v)r^{1-2\alpha}$ are given by:

$$\frac{dv}{dr} = \frac{2r}{r - C(v) - D(v)r^{1-2\alpha}} \ . \tag{22}$$

Now by putting that at the boundary of collapsing cloud $r = r_b$:

$$C(v) + D(v)r_b^{1-2\alpha} = M = const \ , \tag{23}$$

by substituting (23) into (22), and by doing the following transformation:

$$dv = dt + \frac{dr}{1 - \frac{2M}{r}} \ , \tag{24}$$

we finally obtain:

$$\frac{dt}{dr} = \frac{1}{1 - \frac{2M}{r}} \ . \tag{25}$$

And we see that (25) coincides with (21).

If we consider the general case, then we should also demand that:

$$\lim_{r \to r_b} (1 - 2\alpha) \frac{D(v)}{r^{2\alpha}} = 0 \ . \tag{26}$$

The condition (26) can be satisfied if $\alpha > 0$ and $r_b >> D(v)$. In particular, we can smoothly match geodesics in Vaydia—de Sitter spacetime to geodesics in the Schwarzschild one.

The condition (26) appears due to the geodesic equation. In fact, that geodesics in Vaidya spacetime involve the first derivative of the mass function $M(v,r)$ with respect to the coordinate r, and it is not equal to zero like in the Schwarzschild case.

If we want the gravitational collapse of generalized Vaidya spacetime to result in a regular black hole, then we should consider three regions of spacetime:

1) The Vaydia—de Sitter spacetime when $0 < r < r_c$, where $r_c < r_b$;
2) The generalized Vaidya spacetime with the mass function $M(v,r) = C(v) + D(v)r^{1-2\alpha}$ when $r_c < r < r_b$;
3) The Schwarzschild spacetime when $r > r_b$;

To match region II to region III, we should satisfy conditions (23) and (26). To match region I to region II, the following conditions must be met:

$$D_i(v)r_c^3 = C_{ii}(v) + D_{ii}(v)r_c^{1-2\alpha}, \ \ 3D_i(v)r_c^2 = (1-2\alpha)D_{ii}(v)r_c^{2\alpha} \ , \\ r_c < r_b \ . \tag{27}$$

Here I and II denote I and II regions respectively. If we are able to satisfy conditions (23), (26) and (27), then the gravitational collapse might result in a regular black hole.

Unfortunately, with the equation of the state $P = \alpha\rho$, where α is a real number, we always have the approximation (26). To remove it, we should consider a more complex equation of the state.

Finally, we provide geodesics in both Vaydia—de Sitter spacetime and generalized Vaidya spacetime with the mass function $M(v,r) = C(v) + D(v)r^{1-2\alpha}$, where $0 < \alpha \le \frac{1}{3}$ in terms of the mass function $M(v,r)$. They are:

$$\frac{d^2v}{d\tau^2}+\frac{M-M'r}{r^2}\left(\frac{dv}{d\tau}\right)^2-r\left(\frac{d\theta}{d\tau}\right)^2-r\sin^2(\theta)\left(\frac{d\varphi}{d\tau}\right)^2=0\,,$$

$$\frac{d^2r}{d\tau^2}+\frac{\left(1-\frac{2M}{r}\right)(M-M'r)+r\dot{M}}{r^2}\left(\frac{dv}{d\tau}\right)^2$$

$$+2\frac{rM'-M}{r^2}\frac{dv}{d\tau}\frac{dr}{d\tau}+(2M-r)\left(\frac{d\theta}{d\tau}\right)^2+(2M-r)\sin^2(\theta)\left(\frac{d\varphi}{d\tau}\right)^2=0\,, \quad (28)$$

$$\frac{d^2\theta}{d\tau^2}+\frac{2}{r}\frac{dr}{d\tau}\frac{d\theta}{d\tau}-\sin(\theta)\cos(\theta)\left(\frac{d\varphi}{d\tau}\right)^2=0\,,$$

$$\frac{d^2\varphi}{d\tau^2}+\frac{2}{r}\frac{dr}{d\tau}\frac{d\varphi}{d\tau}+2ctg(\theta)\frac{d\varphi}{d\tau}\frac{d\theta}{d\tau}=0\,.$$

For example, if we consider only two regions (II and III) and the result of the gravitational collapse is a naked singularity, then a non-spacelike future-directed geodesic which originates at the singularity in the past crosses the boundary of the collapsing cloud and turns out to be a white hole geodesic in Schwarzschild spacetime.

Conclusion

In this paper, we considered the form of the energy-momentum tensor in the case of a nacked singularity formation. We found out that a naked singularity can be formed only if type II of the matter field dominates. As soon as the matter field of type I appears, the apparent horizon is formed and the singularity stops to be naked.

In order to consider the nature of white hole geodesics, we also considered the Vaydia—de Sitter metric and found out that we can match it to the Schwarzschild one. In order to obtain a regular black hole as a result of such a gravitational collapse, we should consider three regions which are described with Vaydia-de Sitter spacetime, generalized Vaidya spacetime with the mass function $M(v,r) = C(v) + D(v)r^{1-2\alpha}$ and Schwarzschild metric. In this case, a geodesic comes from the infinity but, due to high negative density inside the collapsing cloud matter, turns back to our universe. If the result of a gravitational collapse is a naked singularity, a non-spacelike future-directed geodesic originates at this singularity, crosses the boundary of the ball of collapsing matter, and turns out to be a white hole geodesic in Schwarzschild spacetime.

Of course, for a more realistic case, we should consider a more complex equation of the state and also take into account the rotation. A gravitational collapse of the matter cloud with small rotation is being considered now

Acknowledgments

The author extends his gratitude to Professor D.Ph.-M.Sc. A. A. Grib and Professor Pankaj Joshi for scientific discussion.

References

Brassel, B. P., Goswami, R., Maharaj, S. D. (2017) Collapsing radiating stars with various equations of state. *Physical Review D*, 95 (12), article 124051. https://www.doi.org/10.1103/PhysRevD.95.124051 (In English)

Brassel, B. P., Maharaj, S. D., Goswami, R. (2017) Diffusive and dynamical radiating stars with realistic equations of state. *General Relativity and Gravitation*, 49 (3), article 37. https://www.doi.org/10.1007/s10714-017-2202-2 (In English)

Dawood, A. K., Ghosh, S. G. (2004) Generating dynamical black hole solutions. *Physical Review D*, 70 (10), article 104010. https://www.doi.org/10.1103/PhysRevD.70.104010 (In English)

Dymnikova, I. (1992) Vacuum nonsingular black hole. *General Relativity and Gravitation*, 24 (3), 235–242. https://doi.org/10.1007/BF00760226 (In English)

Dymnikova, I. (2002) The cosmological term as a source of mass. *Classical and Quantum Gravity*, 19 (4), 725–740. https://www.doi.org/10.1088/0264-9381/19/4/306 (In English)

Grib, A. A., Pavlov, Yu. V. (2015) Are black holes totally black? *Gravitation and Cosmology*, 21 (1), 13–18. https://www.doi.org/10.1134/S0202289315010065 (In English)

Grib, A. A., Pavlov, Yu. V., Vertogradov, V. D. (2014) Geodesics with negative energy in the ergosphere of rotating black holes. *Modern Physics Letters A*, 29 (20), article 1450110. https://www.doi.org/10.1142/S0217732314501107 (In English)

Hawking, S. W., Ellis, G. F. R. (1973) *The large scale structure of space-time*. London: Cambridge University Press, 385 p. https://doi.org/10.1017/CBO9780511524646 (In English)

Joshi, P. S. (2007) *Gravitational collapse and spacetime singularities*. Cambridge: Cambridge University Press, 269 p. https://doi.org/10.1017/CBO9780511536274 (In English)

Joshi, P. S., Malafarina, D. (2011) Recent developments in gravitational collapse and spacetime singularities. *International Journal of Modern Physics D*, 20 (14), 2641–2729. https://www.doi.org/10.1142/S0218271811020792 (In English)

Mkenyeleye, M. D., Goswami, R., Maharaj, S. D. (2014) Gravitational collapse of generalized Vaidya spacetime. *Physical Review D*, 90 (6), article 064034. https://www.doi.org/10.1103/PhysRevD.90.064034 (In English)

Nolan, B. C. (1999) Strengths of singularities in spherical symmetry. *Physical Review D*, 60 (2), article 024014. https://www.doi.org/10.1103/PhysRevD.60.024014 (In English)

Oppenheimer, J. R., Snyder, H. (1939) On continued gravitational contraction. *Physical Review*, 56 (5), 455–459. https://www.doi.org/10.1103/PhysRev.56.455 (In English)

Papapetrou, A. (1985) Formation of singularity and causality. In: N. Dadhich (ed.). *A random walk in relativity and cosmology: Essays in honour of P C. Vaidaya and A. K. Raychaudhuri*. New York: Wiley Publ., pp. 184–191. (In English)

Poisson, E. (2004) *A relativist's toolkit: The mathematics of black-hole mechanics.* Cambridge: Cambridge University Press, 233 p. https://doi.org/10.1017/CBO9780511606601 (In English)

Tipler, F. J. (1977) Singularities in conformally flat spacetimes. *Physics Letters A*, 64 (1), 8–10. https://www.doi.org/10.1016/0375-9601(77)90508-4 (In English)

Vertogradov, V. D. (2015) Geodesics for particles with negative energy in Kerr's metric. *Gravitation and Cosmology*, 21 (2), 171–174. https://www.doi.org/10.1134/S0202289315020115 (In English)

Vertogradov, V. D. (2016) Naked singularity formation in generalized Vaidya space-time. *Gravitation and Cosmology*, 22 (2), 220–223. https://www.doi.org/10.1134/S020228931602016X (In English)

Wang, A., Wu, Y. (1999) LETTER: Generalized vaidya solutions. *General Relativity and Gravitation*, 31 (1), 107–114. https://www.doi.org/10.1023/A:1018819521971 (In English)