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The forces and Penrose process in Friedman spacetime

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Abstract. In this paper, we compare the force expressions in both comoving synchronous and Ellis coordinate systems in Friedman-Robertson-Walker spacetime in order to understand how the coordinate transformation affects the inertial forces. We also studied the negative energy problem in Ellis system. The Penrose process is possible outside the apparent horizon due to the fact that the line element has off-diagonal term. However, only the movement towards infinity is possible; we offer the way how one can ascertain that this effect took place outside the apparent horizon.

Keywords: FRW metric, inertial forces, Ellis system, apparent horizon, geodesics, negative energy, Penrose process.

Introduction

In order to describe gravitational force in General Relativity one must study the equations of geodesic lines (Chandrasekhar 1998; Landau, Lifshitz 1980; Weinberg 1972) in the given metric described by some known metric tensor. For example, the leading geodesic equation term in Schwarzschild metric is the Newton force of attraction (Vladimirov 2009). But which are the relevant forces in cosmology as described in the standard Friedman model of the expanding Universe? To answer this question one must write the equations for geodesic lines in Friedmann metric and look for some interpretation of the Christoffel symbols in terms of forces. The form of these symbols is, however, different form in different coordinate systems (Hartle 2002). This occurs because the Christoffel symbols don't obey the tensor law of transformation, which leads to the so called inertial forces being different in different frames. Here, we consider two frames: the comoving synchronous frame and what can be called the Ellis frame (Weinberg 2008). Both frames are widely used in cosmology. The second frame has a more physical appearance, because at small distances it leads to Minkowsky spacetime and only reflects the expansion of the Universe at large distances.This corresponds to the fact that we being on the Earth and in Solar system are not expanding in spite of the fact that on large distances galaxies are expanding. Nevertheless, a synchronous system is well-suited to describe properties of galaxies. In the papers by A. A. Grib and Yu. V. Pavlov (Grib, Pavlov 2021) the radial geodesic movement was studied for these two frames. Here, we shall study the geodesics for angles when the value of the angular momentum is conserved. The structudre of the paper is as folows: first we study the geodesic in the synchronous system, then in the Ellis system and discuss the difference of forces in these two systems.

In the Ellis frame, the metric has the off-diagonal term so one might expect the particles with negative energy to exist outside the apparent horizon. However, the radial movement is only possible towards infinity and there is a question how one can ascertain that the Penrose process occurs outside the apparent horizon. Here, we offer a method that makes it possible to ascertain whether or not this particle possesed negative energy outside the apparent horizon by estimating the positive energy in comoving frame.

We will write some formulas with m and c to emphasize dimensions but in most of formulas, for simplicity, $G = c = 1$ system of units will be used.

The forces in FRW metric

The Friedman-Robertson-Walker (FRW) metric in the coordinates $\{t, r, \theta, \varphi\}$ has the following form:

$$ds^2 = c^2dt^2 - a^2(t)\left[\frac{dr^2}{1-kr^2} - r^2 d\Omega^2\right]. \tag{1}$$

Here a(t) is the scale factor, k = 0, ±1 corresponds the flat, closed and open Friedman model respectively and $d\Omega^2$ is given by:

$$d\Omega^2 = d\theta^2 + sin^2\theta d\varphi^2. \tag{2}$$

It is useful to rewrite the metric (1) in the coordinates $\{t, \chi, \theta, \varphi\}$. For this purpose let us make the following transformation:

$$r = f(\chi), \tag{3}$$

where

$$\begin{aligned} &f(\chi) = sin\chi, \ for\ k = 1,\\ &f(\chi) = \chi, \ for\ k = 0,\\ &f(\chi) = sinh\chi, \ for\ k = -1. \end{aligned} \tag{4}$$

Then one obtains the metric in the new coordinates:

$$ds^2=c^2dt^2 - a^2(t)[d\chi^2+f^2(\chi)d\Omega^2]. \tag{5}$$

The f_χ (and f_d see below) force component has been computed by Yu. Pavlov and A. Grib (Grib, Pavlov 2021). The metric (1) possesses spherical symmetry. It means we can choose any plane and only consider movement in this plane. For simplicity let us pick the equatorial plane, for which $\theta = \frac{\pi}{2}$. It is convenient to express the φ force component through the angular momentum L. We can derive the expression for L from the lagrangian L:

$$2\mathcal{L} = mc^2\left(\frac{dt}{d\tau}\right)^2 - ma^2(t)\left[\left(\frac{d\chi}{d\tau}\right)^2 + f^2(\chi)\left(\frac{d\varphi}{d\tau}\right)^2\right], \tag{6}$$

$$\theta = \frac{\pi}{2}, \ d\theta = 0,$$

where m is the mass of particle, τ is proper time.

From now on we will use the system of units $M = c = 1$.

Using (6) we can obtain the angular momentum expression:

$$L = -\frac{d\mathcal{L}}{d\dot{\varphi}} = a^2(t) f^2(\chi)\dot{\varphi}. \tag{7}$$

Differentiating (7) with respect to τ we obtain the φ component of acceleration:

$$\frac{d^2\varphi}{d\tau^2} = -\frac{L}{a^4(t)f^4(\chi)}\left(2f'(\chi)f(\chi)a^2(t)\frac{d\chi}{d\tau} + 2a(t)\dot{a}(t)f^2(\chi)\frac{dt}{d\tau}\right), \tag{8}$$

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here dash and overdot mean the partial derivative with respect to χ and t respectively. From (7) one can see that the angular momentum depends on the ϕ component of four-velocity $\frac{d\varphi}{d\tau}$. Hence the following expression:

$$-2L\frac{H}{a^2 f^2}\frac{dt}{d\tau}, \tag{9}$$

corresponds to the force which acts like the Coriolis force. Here $H = \frac{\dot{a}}{a}$ is the Hubble constant. This force is attractive if $L < 0$ and repulsive if $L > 0$.

But in (8) we have the second term:

$$-2L\frac{f'}{a^2 f^3}\frac{d\chi}{d\tau}, \tag{10}$$

which has the squared dependence on the velocity. This term doesn't correspond to any force due to the fact that when we consider the inertial forces we should take the three covariant derivative instead of the usual one and terms which are proportional to $\frac{du^\alpha}{d\tau}\frac{du^\beta}{d\tau}$ are the part of this three covariant derivative.

Looking into the force (9), one can see that it is in direct ratio to the angular momentum and the Hubble constant and in inverse ratio to f^2.

Now let us find the difference in the ϕ force component in the case of $\{t, D, \theta, \phi\}$. For this purpose let us make the following transformation:

$$D = a(t)\chi\,. \tag{11}$$

Again we restore c and m for a few formulas.

In these coordinates the metric is given by:

$$ds^2 = \left(1 - \frac{H^2 D^2}{c^2}\right)c^2 dt^2 + 2\frac{HD}{c}dDdct - dD^2 - a^2(t) f^2\left(\frac{D}{a}\right)d\Omega^2\,. \tag{12}$$

In these coordinates we have off-diagonal term $2\frac{HD}{c}dDdct$. The Hubble constant H is positive so when $1 - H^2 D^2$ is less than zero the line element might be timelike due to this off-diagonal term. The surface $D = \frac{c}{H}$ is called the apparent horizon. The region $D = \frac{1}{H} \le D \le +\infty$ has some common features with a black hole. The only radial movement which is allowed is towards infinity. The movement in this region towards the observer at $D = 0$ is forbidden. We will show this in the section below.

Like we did in the previous case, we will obtain the force expression differentiating the angular momentum expression. Again we put the lagrangian in the metric (12) in the following form:

$$2\mathcal{L} = mc^2\left(1 - \frac{H^2 D^2}{c^2}\right)\left(\frac{dt}{d\tau}\right)^2 + 2mc\frac{HD}{c}\frac{dt}{d\tau}\frac{dD}{d\tau} - ma^2(t) f^2\left(\frac{D}{a(t)}\right)\left(\frac{d\varphi}{d\tau}\right)^2, \tag{13}$$

$$\theta = \frac{\pi}{2}\,.$$

And the angular momentum L is given by (again we put $m = c = 1$):

$$L = -\frac{d\mathcal{L}}{d\dot{\varphi}} = a^2(t) f^2\left(\frac{D}{a(t)}\right)\dot{\varphi}\,. \tag{14}$$

Now differentiating with respect to τ one can obtain the geodesic equation:

$$\frac{d^2\varphi}{d\tau^2} = -2\frac{L}{a^4(t)f^4\left(\frac{D}{a(t)}\right)}$$
$$\left(\dot{a}(t)a(t)f^2\left(\frac{D}{a(t)}\right)\frac{dt}{d\tau} + \dot{f}\left(\frac{D}{a(t)}\right)f\left(\frac{D}{a(t)}\right)a^2(t)\frac{dt}{d\tau} + \right. \tag{15}$$
$$\left. f'\left(\frac{D}{a(t)}\right)f\left(\frac{D}{a(t)}\right)a^2(t)\frac{dD}{d\tau}\right).$$

One can note that—in contrast to the previous case—here we have three terms. Two of these have the linear dependence from velocity:

$$-2L\left(\frac{H}{a^2f^2} + \frac{\dot{f}}{a^2f^3}\right)\frac{dt}{d\tau}, \tag{16}$$

and one term has the squared dependence on the velocity:

$$-2L\frac{f'}{a^2f^3}\frac{dD}{d\tau}. \tag{17}$$

One can note that the part of three covariant derivative (17) has the same expression as in the previous case [SF (10)]. The ϕ force component (16), however, merits thorough consideration.

Let us note that:

$$\dot{f}\left(\frac{D}{a(t)}\right) = \frac{df}{d\frac{D}{a}}\frac{-DH}{a(t)}. \tag{18}$$

Substituting this into (16) one can obtain:

$$-2L\frac{H}{a^2f^2}\frac{dt}{d\tau}\left(1 + \frac{\frac{df}{d\frac{D}{a}}}{af}\right). \tag{19}$$

Now if we consider the new Hubble constant:

$$\tilde{H} = H\left(1 + \frac{\frac{df}{d\frac{D}{a}}}{af}\right), \tag{20}$$

then we can write the well-known force expression (7):

$$f^{\varphi}_{cor} = -2L\frac{\tilde{H}}{a^2f^2}. \tag{21}$$

One should note that $\frac{dt}{d\tau} > 0$ because we have the condition of future movement in time, $\tilde{H} > 0$ so as in the previous case this force component might be both attractive and repulsive depending on the sign of the angular momentum L.

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So if we change the coordinate system, then the force components also changes. Thus the observer should be careful not to mix up $\tilde{H}$ and H.

The negative energy problem

If the metric expression has off-diagonal term then the spacetime can contain particles with negative energy. The energy expression can be obtained from the lagrangian (13):

$$E' = \frac{d\mathcal{L}}{d\dot{t}} = \left(1 - H^2 D^2\right)\dot{t} + HD\dot{D} \,. \tag{22}$$

One can see that the first term becomes negative outside the apparent horizon. However, the metric (12) might still be timelike due to off-diagonal term $2HDdDdt$. For timelike geodesic we have one condition $g_{ik}\frac{dx^l}{d\tau}\frac{dx^k}{d\tau} = 1$. In particular it means that $2L = 1$ but outside the apparent horizon it is possible only if $\frac{dD}{d\tau} > 0$. Otherwise we have spacelike geodesic and the particle on this geodesic has the velocity which is bigger than the speed of light and this situation is unphysical.

So the main question we have to answer is how we can observe these particles with negative energy. This type of particles can only form outside the apparent horizon and all particles then move away from the observer. However the apparent horizon is not static one, i.e. it is dynamic. The observer has the same distance for it. So if the four-velocity $\frac{dD}{d\tau}$ of the observer is bigger then the same component of four-velocity of the particle with negative energy, then after this particle can sometimes be seen by observer. However, inside the apparent horizon this particle can't possess the negative energy because $1 - H^2D^2 > 0$ in this region and the energy which the observer measures is positive. To understand whether or not these particles possessed the negative energy outside the apparent horizon we should compare the energy (22) to the energy from the lagrangian (6). Thus, we need to see how the energy depends upon the coordinate transformation. According to the rule of the vector transformation we have:

$$p_i' = \frac{dx^i}{dx'^k} p_k \,, \tag{23}$$

where $p_i = \frac{d\mathcal{L}}{d\dot{x}^i}$ is the four-momentum vector. This rule gives:

$$E' = E - \chi H p_\chi \,, \tag{24}$$

where

$$E = \frac{dt}{d\tau} \,, \qquad p_\chi = \frac{d\chi}{d\tau} \,. \tag{25}$$

So if the value of $\chi H \dot{\chi}$ is bigger than E then the particle possessed negative energy outside the apparent horizon. Crucially, we evaluate the term $\chi H \dot{\chi}$ outside the apparent horizon.

Conclusion

In this paper we have considered the FRW spacetime in two coordinate systems $\{t, \chi, \phi, \theta\}$ and $\{t, D, \phi, \theta\}$ in order to find out how forces in this spacetime depend upon coordinate transformation for ϕ-force component. We have found that in $\{t, \chi, \phi, \theta\}$ coordinates we only have one expression for the ϕ force component but in $\{t, D, \phi, \theta\}$ we have 2 expressions. Thus, when measuring the Hubble constant one should realize which one is observed, not to confuse the real H Hubble constant with $\tilde{H}$ one. One should also note that this ϕ force component can be attractive $L < 0$ or repulsive $L > 0$, a fact that is shared by both coordinate frames.

We have also considered the negative energy problem. The Penrose process exists outside the apparent horizon but due to the fact that only movement towards infinity is possible one needs to find a way to observe it. We found that if the velocity of the observer $\left(\frac{dD}{d\tau}\right)_{obs}$ is bigger than the velocity of the particle with negative energy $\left(\frac{dD}{d\tau}\right)_{nep}$ then this particle can appear inside the apparent horizon possessing positive energy. Then the observer can measure its positive energy and using formula (24) in the comoving synchronous frame they can estimate its value outside the apparent horizon tracing along the geodesic the term $\chi H p_\chi$. Using $D = a(t)\chi$ and the fact that D must be bigger than $\frac{H^2 D^2}{c^2}$ the observer can say wether or not this energy was negative outside the apparent horizon.

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