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In- and out-states of scalar particles confined between two capacitor plates

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Abstract. In this article, a non-commutative integration method of linear differential equations is used to consider the Klein-Gordon-Fock equation with the L-constant electric field with large L, and light cone variables are used to find new complete sets of its exact solutions. These solutions can be related by integral transformations to previously known solutions described by Gavrilov and Gitman (2016b). Then, using the general theory developed by Gavrilov and Gitman (2016a), this article constructs (in terms of new solutions) the so-called in- and out-states of scalar particles confined between two capacitor plates.

Keywords: exact solutions, Klein-Gordon-Fock equation, potential step, strong electric field, non-commutative integration method.

Introduction

Particle production from vacuum by strong electric-like external fields—the Schwinger effect (Schwinger 1951) or the effect of the vacuum instability—is one of the most interesting effects in quantum field theory (QFT) that scientists have already been researching for a long time. The effect can be observable if the external fields are sufficiently strong, e.g. the magnitude of an electric field should be comparable with the Schwinger critical field with $E_c = m^2c^3/e\hbar \simeq 10^{16}$ V/cm. Nevertheless, recent progress in laser physics brings hope that an experimental observation of the effect can become possible in the near future (for the review, see Dunne 2009; 2014; Di Piazza, Müller, Hatsagortsyan et al. 2012; Mourou, Tajima 2014; Hegelich, Mourou, Rafelski 2014). Moreover, electron-hole pair creation from vacuum also becomes observable in laboratory conditions in graphene and similar nanostructures (see, e.g. Sarma, Adam, Hwang et al. 2011; Vafek, Vishwanath 2014). Various approaches have been proposed for calculating the effect depending on the structure of such external backgrounds (for a list of relevant publications see Ruffini, Vereshchagin, Xue 2010; Gelis, Tanji 2016). Calculating quantum effects in strong external backgrounds must be non-perturbative with respect to the interaction with strong backgrounds. A general formulation of QED with time-dependent external fields (the so-called t-potential steps) was developed by Gitman (1977), Fradkin, Gitman (1981), and Fradkin, Gitman, Shvartsman (1991). It can be also seen that in some situations the vacuum instability effects in graphene and similar nanostructures caused by strong (with respect of massless fermions) electric fields are of significant interest (see Gelis, Tanji 2016; Allor, Cohen, McGady 2008; Gavrilov, Gitman, Yokomizo 2012; Vafek, Vishwanath 2014; Kané, Lazzeri, Mauri 2015; Oladyshkin, Bodrov, Sergeev et al. 2017; Akal, Egger, Müller et al. 2019 and references therein). At the same time, in these cases electric fields can be considered as time-independent weakly inhomogeneous x-electric potential steps (electric fields of constant direction that are concentrated in restricted space areas) that can be approximated by a linear potential. Approaches for treating quantum effects in the explicitly time-dependent external fields are not directly applicable to the x-electric potential steps. A consistent non-perturbative formulation of QED with critical x-electric potential steps, strong enough to violate the vacuum stability, was constructed in the recent work (Gavrilov, Gitman 2016a). A non-perturbative calculation technique for different quantum processes, such as scattering, reflection, and electron-positron pair creation, was developed there. This technique essentially uses special sets of exact solutions of the Dirac and Klein-Gordon equation with the corresponding external field of x-electric potential steps. The cases when such solutions can be found explicitly (analytically) are called exactly solvable cases. This technique was effectively used to describe particle creation effect in the Sauter field of the form $E(x) = E\cosh^{-2}(x/L_S)$, in a constant electric field between two capacitor plates separated by a distance L (the so-called L-constant electric field), and in exponential time-independent electric steps, where the corresponding exact solutions were available (see Gavrilov, Gitman 2016a; 2016b; Gavrilov, Gitman, Shishmarev 2017). These exactly solvable models make it possible to develop a new approximate calculation method to non-perturbatively treat the vacuum instability in arbitrary weakly-inhomogeneous x-electric potential steps (Gavrilov, Gitman, Shishmarev 2019). Note that the corresponding limiting case of a constant uniform electric field shares many similarities with the case of the de Sitter background (see Anderson, Mottola 2014; Akhmedov, Popov 2015 and references therein). Thus, a study of the vacuum instability in the presence of the L -constant electric field with large $L \to \infty$ may be quite important for some applications. Only a critical step with a potential difference $\Delta U > 2m$ (where m is the electron mass) can produce electron-positron pairs; moreover, pairs are born only with quantum numbers in a finite range—in the so-called Klein zone.

As a matter of fact, non-perturbative calculation techniques are related to the possibility of constructing exact solutions of the corresponding relativistic Dirac and Klein-Gordon equations; for instance, solutions that have special asymptotics. Constructing such solutions is a rather difficult task. An adequate choice of variables in the corresponding equations can be useful to solve it. For instance, Narozhnyi and Nikishov (1976) found the above-mentioned solutions in a special representation considering the Dirac and Klein-Gordon equations with a constant uniform field given by time-dependent potential and choosing the variables of the light cone (see also Gavrilov, Gitman, Shvartsman 1979; Gavrilov, Gitman, Gonçalves 1998). These solutions make it possible to explicitly find all kinds of the corresponding QED singular functions in the Fock-Schwinger proper time representation. This present article uses a non-commutative integration method of linear differential equations to consider the Klein-Gordon equation with the L-constant electric field with a large L and uses the light cone variables to find new complete sets of its nonstationary exact solutions. These solutions can be related by integral transformations to previously known stationary solutions that were found by Gavrilov and Gitman (2016b). Then, the general theory developed by Gavrilov and Gitman (2016a) is used to construct—in terms of the new nonstationary solutions—the so-called in- and out-states of scalar particles confined between two capacitor plates.

In- and out-solutions

Let us construct in- and out-solutions of the Klein-Gordon equation with an external constant electric field, which is the so-called L-constant electric field and belongs to the class of x-potential steps. The equation has the form

$$\begin{gathered}\left(P^{\mu}P_{\mu}-m\right)\psi(X)=0,\ P_{\mu}=i\partial_{\mu}-qA_{\mu}(X),\\ P^{\mu}=\eta^{\mu\nu}P_{\nu},\ \eta^{\mu\nu}=\mathrm{diag}\,\underbrace{(1,-1,\dots,-1)}_{d},\ d=D+1,\end{gathered} \tag{1}$$

where $A_\mu(X)$ are corresponding electromagnetic potentials, m is the particle mass and $q = -e,\ e > 0$ is its charge. For the purpose of generality, the problem is considered in d-dimensional spacetime ($\hbar = c = 1$). Here $X = (X^\mu) = (t, \mathbf{r})$, $\mathbf{r} = (X^k)$, $\mu = 0,1, \dots D,\ k = 1, \dots, D$. The L-constant electric field $E(x)$ has the form

$$E(x)=\begin{cases}0, & x\in(-\infty,-L/2]\cup[L/2,\infty)\\ E, & x\in(-L/2,L/2)\end{cases},\quad L>0. \tag{2}$$

We assume that the corresponding x-potential step is critical and sufficiently large, so that $eEL \gg 2m$. In this case, the field $E(x)$ and the leading contributions to the vacuum mean values can be considered as macroscopic ones. However, this L-constant electric field is weakly inhomogeneous, the corresponding Klein zone is extensive, so that all the universal properties of the vacuum instability described by Gavrilov, Gitman and Shishmarev (2019) hold true. The L-constant field in the limit $L\to\infty$ is a kind of a regularisation for a constant uniform electric field. In fact, the L-constant field may be approximated in this limit by a constant uniform electric field given by a linear potential

$$A_0(X)=-Ex,\quad A_k(X)=0,\quad x=X^1,\quad E>0. \tag{3}$$

Consider stationary solutions of the Klein-Gordon equation with the following form:

$$\begin{aligned}&\psi_n(X)=\varphi_n(t,x)\varphi_{\mathbf{p}_\perp}(\mathbf{r}_\perp),\quad \varphi_{\mathbf{p}_\perp}(\mathbf{r}_\perp)=(2\pi)^{-(d-1)/2}\exp(i\mathbf{p}_\perp\mathbf{r}_\perp),\\ &\varphi_n(t,x)=\exp(-ip_0t)\varphi_n(x),\quad n=(p_0,\mathbf{p}_\perp),\\ &X=(t,x,\mathbf{r}_\perp),\mathbf{r}_\perp=(X^2,\dots,X^D),\ \mathbf{p}_\perp=(p^2,\dots,p^D),\ \hat{p}_x=-i\partial_x.\end{aligned} \tag{4}$$

These solutions are quantum states of spinless particles with given energy p_0 and momenta $\mathbf{p}_\perp$ perpendicular to the x-direction. The functions $\varphi_n(x)$ obey the second-order differential equation

$$\{\hat{p}_x^2-[p_0-U(x)]^2+\mathbf{p}_\perp^2+m^2\}\varphi_n(x)=0,\quad U(x)=-eA_0(x). \tag{5}$$

Let us construct two complete sets of solutions with the form (4) and denote them as ${}_\zeta\psi_n(X)$ and ${}^\zeta\psi_n(X)$, $\zeta=\pm$ with special left and right asymptotics:

$$\hat{p}_x\,{}_\zeta\psi_n(X)=p^{\mathrm{L}}\,{}_\zeta\psi_n(X),\quad x\to-\infty,$$

$$\hat{p}_x\,{}^\zeta\psi_n(X)=p^{\mathrm{R}}\,{}^\zeta\psi_n(X),\quad x\to+\infty.$$

The solutions ${}_\zeta\psi_n(X)$ and ${}^\zeta\psi_n(X)$ asymptotically describe particles with given real momenta $p^{\mathrm{L/R}}$ along the x direction. The corresponding functions $\varphi_n(x)$ are denoted by ${}_\zeta\varphi_n(x)$ and ${}^\zeta\varphi_n(x)$, respectively. These functions have the asymptotics

$${}_\zeta\varphi_n(x)={}_\zeta\mathcal{N}\exp\left[i\left|p^{\mathrm{L}}\right|x\right],\quad x\to-\infty,$$

$${}^\zeta\varphi_n(x)={}^\zeta\mathcal{N}\exp\left[i\left|p^{\mathrm{R}}\right|x\right],\quad x\to+\infty.$$

Solutions ${}_\zeta\psi_n(X)$ and ${}^\zeta\psi_n(X)$ are subjected to the following orthonormality conditions with respect to the Klein-Gordon inner product on the $x=$ const hyperplane:

$$\begin{aligned}&\left({}_\zeta\psi_n,{}_{\zeta'}\psi_{n'}\right)_x=\left({}^\zeta\psi_n,{}^{\zeta'}\psi_{n'}\right)_x=\zeta\delta_{\zeta,\zeta'}\delta_{n,n'},\\ &\delta_{n,n'}=\delta(p_0-p_0')\delta(\mathbf{p}_\perp-\mathbf{p}_\perp'),\\ &(\psi,\psi')_x=i\int\psi^*(X)\left(\overleftarrow{\partial}_x-\overrightarrow{\partial}_x\right)\psi'(X)dtd\mathbf{r}_\perp.\end{aligned} \tag{6}$$

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Note that for two solutions with different quantum numbers n, the inner product $(\psi,\psi')_x$ can be easily calculated as

$$\left(\psi_n,\psi'_{n'}\right)_x = \mathcal{I}\,(2\pi)^{d-1}\delta_{n,n'},\ \mathcal{I} = \varphi_n^*(x)\left(i\overleftarrow{\partial}_x - i\overrightarrow{\partial}_x\right)\varphi'_n(x). \tag{7}$$

Solutions ${}_\zeta\psi_n(X)$ and ${}^\zeta\psi_n(X)$ can be decomposed through each other as follows:

$$\begin{aligned} {}^\zeta\psi_n(X) &= {}_+\psi_n(X)g\left({}_+\middle|{}^\zeta\right) - {}_-\psi_n(X)g\left({}_-\middle|{}^\zeta\right), \\ {}_\zeta\psi_n(X) &= {}^-\psi_n(X)g\left({}^-\middle|{}_\zeta\right) - {}^+\psi_n(X)g\left({}^+\middle|{}_\zeta\right), \end{aligned} \tag{8}$$

where the expansion coefficients are defined by the equations

$$\left({}_\zeta\psi_n,\ {}^{\zeta'}\psi_{n'}\right)_x = g\left({}_\zeta\middle|{}^{\zeta'}\right)\delta_{n,n'},\quad g\left({}^{\zeta'}\middle|{}_\zeta\right) = g\left({}_\zeta\middle|{}^{\zeta'}\right)^*.$$

Equation (5) can be written as

$$\left[\frac{\mathrm{d}^2}{\mathrm{d}\xi^2} + \xi^2 - \lambda\right]\varphi_n(x) = 0,\quad \xi = \frac{eEx - p_0}{\sqrt{eE}},\quad \lambda = \frac{\pi_\perp^2}{eE}.$$

Its general solution can be written in terms of an appropriate pair of linearly independent Weber parabolic cylinder functions (WPCFs), either as $D_\rho[(1-\mathrm{i})\xi]$ and $D_{-1-\rho}[(1+\mathrm{i})\xi]$ or $D_\rho[-(1-\mathrm{i})\xi]$ and $D_{-1-\rho}[-(1+\mathrm{i})\xi]$, where $\rho = -\mathrm{i}\lambda/2 - 1/2$.

Using asymptotic expansions of WPCFs, the functions ${}_\zeta\varphi_n(x)$ and ${}^\zeta\varphi_n(x)$ can be constructed as

$$\begin{aligned} {}_+\varphi_n(x) &= {}_+\mathcal{N}D_{-1-\rho}[-(1+\mathrm{i})\xi] \sim e^{-i\xi^2/2},\quad \xi\to-\infty,\ p^{\mathrm{L}} = -\xi\sqrt{eE} \\ {}_-\varphi_n(x) &= {}_-\mathcal{N}D_\rho[-(1-\mathrm{i})\xi] \sim e^{i\xi^2/2},\quad \xi\to-\infty,\ p^{\mathrm{L}} = \xi\sqrt{eE}; \\ {}^+\varphi_n(x) &= {}^+\mathcal{N}D_\rho[(1-\mathrm{i})\xi] \sim e^{i\xi^2/2},\quad \xi\to\infty,\ p^{\mathrm{R}} = \xi\sqrt{eE}, \\ {}^-\varphi_n(x) &= {}^-\mathcal{N}D_{-1-\rho}[(1+\mathrm{i})\xi] \sim e^{-i\xi^2/2},\quad \xi\to\infty,\ p^{\mathrm{R}} = -\xi\sqrt{eE}, \end{aligned} \tag{9}$$

$${}_\zeta\mathcal{N} = {}^\zeta\mathcal{N} = (2eE)^{-1/4}e^{\pi\lambda/8}. \tag{10}$$

Their in- and out-classifications are related to the signs of the asymptotic momenta p^{L} and p^{R} (see Gavrilov, Gitman 2016b). Namely,
${}_+\psi_n,\ {}^+\psi_n$ are in-states and ${}_-\psi_n,\ {}^-\psi_n$ are out-states.

It is useful to construct two different complete sets of solutions of the Klein-Gordon equation (1) that are not stationary states and can be written as

$$\psi_\sigma(X) = \varphi_\sigma(t,x)\varphi_{\mathbf{p}_\perp}(\mathbf{r}_\perp), \tag{11}$$

where σ is a set of quantum numbers that will be defined below. In this case, the function $\varphi_\sigma(t,x)$ satisfies the equation

$$\{\hat{p}_x^2 - [\hat{p}_0 - U(x)]^2 + \mathbf{p}_\perp^2 + m^2\}\varphi_\sigma(t,x) = 0,\quad \hat{p}_0 = i\partial_t. \tag{12}$$

This equation admits integrals of motion in the class of linear differential operators of the first order, which are

$$\hat{Y}_0 = -ie,\quad \hat{Y}_1 = \partial_t,\quad \hat{Y}_2 = \partial_x + ieEt,\quad \hat{Y}_3 = x\partial_t + t\partial_x + \frac{ieE}{2}(t^2 + x^2).$$

The operators $\hat{Y}_a$, $a = 0,1,2,3$ form a four-dimensional Lie algebra $\mathfrak{g}$ with nonzero commutation relations

$$[\hat{Y}_1, \hat{Y}_2] = -E\,\hat{Y}_0, \quad [\hat{Y}_1, \hat{Y}_3] = \hat{Y}_2, \quad [\hat{Y}_2, \hat{Y}_3] = \hat{Y}_1.$$

Equation (12) can be considered as an equation for the eigenfunctions of the Casimir operator $K(-i\hat{Y}) = -2E\,\hat{Y}_0\hat{Y}_3 + \hat{Y}_1^2 - \hat{Y}_2^2$,

$$-K(-i\hat{Y})\varphi_\sigma(t,x) = (\mathbf{p}_\perp^2 + m^2)\varphi_\sigma(t,x).$$

At this stage, we follow a non-commutative integration method of linear differential equations (Shapovalov, Shirokov 1995; Bagrov, Baldiotti, Gitman et al. 2002; Breev, Shapovalov 2016), which allows us to construct a complete set of solutions based on a symmetry of the equation. An irreducible representation of the Lie algebra $\mathfrak{g}$ in the space of functions of the variable $\tilde{p} \in (-\infty, +\infty)$ is defined by the help of the operators $\ell_a(\tilde{p}, \partial_{\tilde{p}}, j)$:

$$\ell_0(\tilde{p}, \partial_{\tilde{p}}, j) = ie, \quad \ell_1(\tilde{p}, \partial_{\tilde{p}}, j) = -eE\partial_{\tilde{p}} + \frac{i}{2}\tilde{p},$$

$$\ell_2(\tilde{p}, \partial_{\tilde{p}}, j) = eE\partial_{\tilde{p}} + \frac{i}{2}\tilde{p}, \quad \ell_3(\tilde{p}, \partial_{\tilde{p}}, j) = -\tilde{p}\partial_{\tilde{p}} + ij - \frac{1}{2}, \quad j > 0,$$

where j parameterises the non-degenerate adjoint orbits of a Lie algebra $\mathfrak{g}$. The following relations hold true:

$$[\ell_1, \ell_2] = -E\,\ell_0, \quad [\ell_1, \ell_3] = \ell_2, \quad [\ell_2, \ell_3] = \ell_1,$$
$$K(-i\ell(\tilde{p}, \partial_{\tilde{p}}, j)) = (2eE)j.$$

Integrating the equations

$$[\hat{Y}_a + \ell_a(\tilde{p}, \partial_{\tilde{p}}, j)]\varphi_\sigma(t,x) = 0 \tag{13}$$

together with the equation (5), we fix $j = -\lambda/2$ and derive a set of solutions which is characterised by quantum numbers $\sigma = (\tilde{p}, \mathbf{p}_\perp)$,

$$\begin{aligned} {}^{\pm}_{-}\varphi_\sigma(t,x) &= {}^{\pm}_{-}C_\sigma \exp\left(ie\frac{E}{2}\left[\frac{1}{2}x_-^2 - t^2\right] - \frac{i}{2}[\lambda - i]\left(\ln\frac{\pm i\pi_-}{\sqrt{eE}}\right) - \frac{i}{2}\tilde{p}\,x_+\right), \\ \pi_- &= \tilde{p} + eEx_-, \quad x_\pm = t \pm x. \end{aligned} \tag{14}$$

The parameter $\tilde{p}$ is an eigenvalue of the symmetry operator $i(\hat{Y}_1 + \hat{Y}_2)$:

$$i(\hat{Y}_1 + \hat{Y}_2)\,{}^{\pm}_{-}\varphi_\sigma(t,x) = \tilde{p}\,{}^{\pm}_{-}\varphi_\sigma(t,x).$$

It is possible to interpret the quantum numbers σ from the perspective of the orbit method: the parameter $\lambda = (m^2 + \mathbf{p}_\perp^2)/(eE)$ describes the Casimir operator $K(-i\hat{Y})$ spectrum and parameterises the non-degenerate orbits of the co-adjoint representation of the local Lie group $\exp\mathfrak{g}$ (in this case, the orbits are hyperbolic paraboloids), and the variation region of the parameter $\tilde{p}$ is a Lagrangian submanifold to these orbits.

In order to classify solutions (11), we define direct and inverse integral transformations that relate these solutions to solutions (4), which are stationary states, eigenfunctions for the operator $\hat{p}_0$.

We represent solutions of both equation (5) and

$$\hat{p}_0\,\varphi_n^{(\pm)}(t,x) = p_0\,\varphi_n^{(\pm)}(t,x)$$

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in the following form

$$\varphi_n^{(\pm)}(t,x) = (2\pi eE)^{-1/2}\int_{-\infty}^{+\infty} M^*(p_0,\tilde{p})\,{}_{-}^{+}\varphi_\sigma(t,x)dp. \tag{15}$$

Taking into account condition (13), the equation for the function $M(p_0,\tilde{p})$ can be written as:

$$-i\ell_1(\tilde{p},\partial_{\tilde{p}},j)M(p_0,\tilde{p}) = p_0 M(p_0,\tilde{p}).$$

We choose its particular solution

$$M(p_0,\tilde{p}) = \exp\left(\frac{i}{4eE}[\tilde{p}^2 - 4\tilde{p}p_0]\right), \tag{16}$$

which satisfies the orthogonality relation

$$\int_{-\infty}^{+\infty} M^*(p_0,\tilde{p})M(p_0,\tilde{p}')dp_0 = 2\pi eE\,\delta(\tilde{p}-\tilde{p}'). \tag{17}$$

The inverse to (15) transformation reads:

$${}_{-}^{+}\varphi_\sigma(t,x) = (2\pi eE)^{-1/2}\int_{-\infty}^{+\infty} M(p_0,\tilde{p})\varphi_n^{(\pm)}(t,x)dp_0. \tag{18}$$

Thus, direct (15) and inverse (18) integral transformation were defined with kernel (16) that converts solutions (14) into solutions that are eigenfunctions for the operator $\hat{p}_0$. Applying one of the integral transformations to solutions (14), we get:

$$\begin{aligned}\varphi_n^{(\pm)}(t,x) &= (2\pi eE)^{-\frac{1}{2}}\int_{-\infty}^{+\infty} M^*(p_0,\tilde{p})\,{}_{-}^{+}\varphi_\sigma(t,x)d\tilde{p} = \\ &= {}_{-}^{+}C_\sigma(1-i)^{\rho+1}e^{\frac{ip_0^2}{2eE}}e^{-ip_0t}D_\rho[\pm(1-i)\xi].\end{aligned} \tag{19}$$

Then, comparing the equation (19) with the equations (9)-(10) gives the following correspondence:

$$\begin{aligned}\varphi_n^{(+)}(t,x) &\sim {}^{+}\mathcal{N}'e^{-ip_0t}D_\rho[+(1-\mathrm{i})\xi] = {}^{+}\varphi_n(x)e^{-ip_0t}, \\ \varphi_n^{(-)}(t,x) &\sim {}_{-}\mathcal{N}'e^{-ip_0t}D_\rho[-(1-\mathrm{i})\xi] = {}_{-}\varphi_n(x)e^{-ip_0t}.\end{aligned} \tag{20}$$

Transformation (18) makes it possible to derive orthonormality relations on the hyperplane $x = \text{const}$ for scalar particles constructing with the help of functions ${}_{-}^{+}\varphi_\sigma(t,x)$,

$$({}_{-}^{+}\psi_\sigma, {}_{-}^{+}\psi_{\sigma'})_x = \pm\delta_{\sigma,\sigma'}, \tag{21}$$

where

$${}_{-}^{+}\psi_\sigma(X) = {}_{-}^{+}\varphi_\sigma(t,x)\varphi_{\mathbf{p}_\perp}(\mathbf{r}_\perp), \tag{22}$$

and determine the normalising factors ${}_{-}^{+}C_\sigma$,

$${}_{-}^{+}C_\sigma = \frac{1}{\sqrt{4\pi eE}}e^{\pi\lambda/4}.$$

Thus, we obtain:

$$({}_{-}\psi_\sigma, {}^{+}\psi_{\sigma'})_x = g({}_{-}|{}^{+})\delta_{\sigma,\sigma'}, \quad g({}_{-}|{}^{+}) = ie^{\pi\lambda/2}.$$

Let us introduce the following notation:

$$\varphi_n^{(\pm)}(t,x) = {}_{-}^{+}\varphi_n(x)\exp(-ip_0t),\ {}_{-}^{+}\psi_n(X) = \varphi_n^{(\pm)}(t,x)\varphi_{\mathbf{p}_\perp}(\mathbf{r}_\perp).$$

It follows from (15) and (18) that

$$\begin{aligned} {}_{-}^{+}\psi_\sigma(X) &= (2\pi eE)^{-1/2}\int_{-\infty}^{+\infty} M(p_0,\tilde{p})\ {}_{-}^{+}\psi_n(X)dp_0, \\ {}_{-}^{+}\psi_n(X) &= (2\pi eE)^{-1/2}\int_{-\infty}^{+\infty} M^*(p_0,\tilde{p})\ {}_{-}^{+}\psi_\sigma(X)d\tilde{p}. \end{aligned} \tag{23}$$

Let us consider another type of solutions:

$$\begin{aligned} {}_{+}\varphi_\sigma(t,x) &= \theta(-\pi_-)\ {}^{+}\varphi_\sigma(t,x), \\ {}^{-}\varphi_\sigma(t,x) &= \theta(+\pi_-)\ {}_{-}\varphi_\sigma(t,x). \end{aligned} \tag{24}$$

The corresponding integral transformation is

$$\begin{aligned} {}_{+}\varphi_n(t,x) &= (2\pi eE)^{-\frac{1}{2}}\int_{-\infty}^{+\infty} M^*(p_0,\tilde{p})\theta(-\pi_-)\ {}^{+}\varphi_\sigma(t,x)d\tilde{p} = \\ &= -\ {}^{+}C_\sigma\sqrt{\frac{2}{\pi}}(-[1+i])^{\rho-1}\,\Gamma(\rho+1)e^{\frac{ip_0^2}{2eE}}e^{-ip_0t}D_{-\rho-1}[-(1+i)\xi] \\ {}^{-}\varphi_n(t,x) &= (2\pi eE)^{-\frac{1}{2}}\int_{-\infty}^{+\infty} M^*(p_0,\tilde{p})\theta(\pi_-)\ {}_{-}\varphi_\sigma(t,x)d\tilde{p} = \\ &= -\ {}_{-}C_\sigma\sqrt{\frac{2}{\pi}}(-[1+i])^{\rho-1}\,\Gamma(\rho+1)e^{\frac{ip_0^2}{2eE}}e^{-ip_0t}D_{-\rho-1}[(1+i)\xi], \end{aligned} \tag{25}$$

so that

$$\begin{aligned} {}_{+}\varphi_n(t,x) &\sim\ {}_{+}\mathcal{N}'e^{-ip_0t}D_{-1-\rho}[-(1+\mathrm{i})\xi] =\ {}_{+}\varphi_n(x)e^{-ip_0t}, \\ {}^{-}\varphi_n(t,x) &\sim\ {}^{-}\mathcal{N}'e^{-ip_0t}D_{-1-\rho}[(1+\mathrm{i})\xi] =\ {}^{-}\varphi_n(x)e^{-ip_0t}. \end{aligned} \tag{26}$$

Using functions ${}_{+}^{-}\varphi_\sigma(t,x)$, we construct a new set of solutions

$${}_{+}^{-}\psi_\sigma(t,x)(X) = \sqrt{e^{\pi\lambda}-1}\ {}_{+}^{-}\varphi_\sigma(t,x)\varphi_{\mathbf{p}_\perp}(\ \mathbf{r}_\perp), \tag{27}$$

which satisfies the following orthonormality relations:

$$(\ {}_{+}\psi_\sigma, {}_{-}\psi_{\sigma'})_x = 0,\ (\ {}^{-}\psi_\sigma, {}^{+}\psi_{\sigma'})_x = 0,\ ({}_{+}^{-}\psi_\sigma, {}_{+}^{-}\psi_\sigma)_x = \pm\delta_{\sigma,\sigma'}.$$

Based on (25) and (17), the integral transformations are

$$\begin{aligned} {}_{+}^{-}\psi_\sigma(X) &= (2\pi eE)^{-1/2}\int_{-\infty}^{+\infty} M(p_0,\tilde{p})\ {}_{+}^{-}\psi_n(X)dp_0, \\ {}_{+}^{-}\psi_n(X) &= (2\pi eE)^{-1/2}\int_{-\infty}^{+\infty} M^*(p_0,\tilde{p})\ {}_{+}^{-}\psi_\sigma(X)d\tilde{p}. \end{aligned} \tag{28}$$

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There exist useful relations between solutions ${}_\zeta\psi_\sigma(X)$ and ${}^\zeta\psi_\sigma(X)$. Each of them is complete for a given σ and can be decomposed through another one as follows:

$$\begin{aligned} {}^\zeta\psi_\sigma(X) &= {}_+\psi_\sigma(X) g({}_+|{}^\zeta) - {}_-\psi_\sigma(X) g({}_-|{}^\zeta), \\ {}_\zeta\psi_\sigma(X) &= {}^-\psi_\sigma(X) g({}^-|{}_\zeta) - {}^+\psi_\sigma(X) g({}^+|{}_\zeta), \end{aligned} \tag{29}$$

Equations

$$\left({}_\zeta\psi_\sigma, {}^{\zeta'}\psi_{\sigma'}\right)_x = g({}_\zeta|{}^{\zeta'})\delta_{\sigma,\sigma'}, \quad g({}^{\zeta'}|{}_\zeta) = g({}_\zeta|{}^{\zeta'})^*$$

allow us to calculate coefficients $g({}_-|{}^-)$ and $g({}_+|{}^+)$,

$$g({}^-|{}_-) = -\sqrt{e^{\pi\lambda} - 1}, \quad g({}^+|{}_+) = +\sqrt{e^{\pi\lambda} - 1}.$$

We note that the relations (29) are similar to the relations (8) that were established for the solutions ${}_\zeta\psi_n(X)$ and ${}^\zeta\psi_n(X)$ (in this case, the coefficients g do not depend on p_0 and $\tilde{p}$). From (20) and (26) it follows that ${}_+\psi_n$, ${}^+\psi_n$ are in-states and ${}_-\psi_n$, ${}^-\psi_n$ are out-states.

From the equation (24) it follows that

$$\begin{aligned} {}_+\psi_\sigma(X) &= 0, \quad \pi_- > 0, \\ {}^-\psi_\sigma(X) &= 0, \quad \pi_- < 0. \end{aligned} \tag{30}$$

Then, taking into account equations (29) and (30), we get:

$$\begin{aligned} {}_+\psi_\sigma(X) &= g({}_+|{}^-)^{-1}[{}_-\psi_\sigma(X) g({}_-|{}^-) + \\ &+ {}^-\psi_\sigma(X)] = 0, \quad \pi_- > 0, \\ {}^-\psi_\sigma(X) &= g({}^-|{}_+)^{-1}[{}^+\psi_\sigma(X) g({}^+|{}_+) + \\ &+ {}_+\psi_\sigma(X)] = 0, \quad \pi_- < 0. \end{aligned} \tag{31}$$

Since the coefficient $g({}_+|{}^-)$ is not zero for all σ, the equations (31) imply a direct connection between the solutions ${}_\zeta\psi_\sigma(X)$ and ${}^\zeta\psi_\sigma(X)$ normalised on the hyperplane $x = \text{const}$,

$$\begin{aligned} {}^-\psi_\sigma(X) &= -{}_-\psi_\sigma(X) g({}_-|{}^-)\theta(\pi_-), \\ {}_+\psi_\sigma(X) &= -{}^+\psi_\sigma(X) g({}^+|{}_+)\theta(-\pi_-). \end{aligned}$$

Thus, using the non-commutative integration method for the equation (12), we obtained in- and out-states of scalar particles in terms of new solutions (22) and (27), which are non-stationary and are determined by a set of quantum numbers σ. Solutions $\{{}_+\psi_\sigma, {}^+\psi_\sigma\}$ describe in-states and solutions $\{{}_-\psi_\sigma, {}^-\psi_\sigma\}$ describe out-states. It follows from the integral transformations (23) and (28) that the solutions ${}_\zeta\psi_\sigma(X)$ and ${}^\zeta\psi_\sigma(X)$ are related to the well-known stationary solutions ${}_\zeta\psi_n(X)$ and ${}^\zeta\psi_n(X)$ (see Gavrilov, Gitman 2016b).

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