

Some remarks on the naked singularity phenomenon

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Abstract. This paper describes a naked singularity phenomenon. The study presents necessary definitions and methods for dealing with this effect. Moreover, the article gives two explicit examples of the naked singularity formation.

Keywords: gravitational collapse, naked singularity, strength of singularity, black hole, Vaidya spacetime.

Introduction

Black holes are among the most appealing fields of modern theoretical physics. In 1916 Schwarzschild solved the Einstein equation and showed that there is a line at where all physical quantities are blown up. This is called a singularity. All famous black hole models have a singularity which is hidden under the event horizon. In 1970, Penrose came up with the cosmic censorship principle (CCP). According to this principle, all singularities must be hidden under the event horizon. However, all these models describe eternal black holes without mentioning anything about their formation. Therefore, in order to understand how a black hole is formed, it is necessary to consider gravitational collapse.

In 1939, Oppenheimer and Snyder (Oppenheimer, Snyder 1939) considered gravitational collapse of the pressureless homogeneous matter cloud. They showed that such a collapse results in a black hole. A black hole was believed to be the only result of gravitational collapse for a long time. However, later it was found that a singularity might be formed earlier than the apparent horizon during gravitational collapse (see Joshi 2007; Goswami, Joshi 2007). An apparent horizon is a marginally trapped surface and a boundary of the trapped region. Thus, if it is absent and there is a family of non-spacelike future-directed geodesics which terminate at the central singularity in the past, then such a singularity is visible and is called a naked singularity. If we consider gravitational collapse of the inhomogeneous dust, then the result of such a collapse might be a naked singularity. It should be noted that naked singularity is temporary, and in a short period of time it is covered with the apparent horizon, which results in a black hole.

Naked singularity formation violates CCP in its original version. However, after 1970 other CCPs were formulated and one of them admits naked singularity formation in four-dimensional spacetime (for details on all CCPs, see Joshi 2007).

When gravitational collapse is considered, vacuum solutions cannot be considered anymore. We must put the energy-momentum tensor (EMT) on the right side of the Einstein equation. The EMT must be physically relevant and must satisfy energy conditions: weak, strong and dominant ones (Hawking, Ellis 1973). Nowadays there are a lot of gravitational collapse models which result in naked singularities (Goswami, Joshi 2007; Vertogradov 2016; 2018; Mkenyeley, Goswami, Maharaj 2014); however, if the EMT satisfies all necessary energy conditions, these naked singularities are temporary. Nevertheless, one en-

ergy condition—the strong one—can be violated. In this case matter distribution is thought to possess negative pressure and an eternal naked singularity might be formed (Goswami, Joshi 2007; Vertogradov 2018). It should be noted that in this case ‘eternal’ means that the singularity is formed during the gravitational collapse process and will never be covered with the apparent horizon.

This paper has the following structure. Section II provides definitions of all necessary terms such as ‘energy condition’, ‘apparent horizon’, ‘expansion’, etc. Section III describes naked singularity formation in the general spherical symmetry case. Section IV gives necessary conditions for the naked singularity formation in case of generalised Vaidya spacetime. Section V presents the violation of the strong energy condition and considers the possibility of eternal naked singularity formation. Section VI is the conclusion. Units $g = c = 1$ are used throughout this paper. Moreover, the “- + + +” signature is used in this paper.

$$M' = \frac{\delta M}{\delta r}, \dot{M} = \frac{\delta M}{\delta t} = \frac{\delta M}{\delta v}$$

are partial derivatives with respect to coordinate r and time t and v , respectively.

Necessary definitions

The energy condition

When gravitational collapse is considered, the EMT is put on the right side of Einstein equations. However, the EMT must obey certain conditions since these models should be physically relevant and should never have negative mass or negative energy density, etc. Therefore, it is possible to impose conditions that energy density must be positive, or that energy density must be more than pressure (Hawking, Ellis 1973; Poisson 2004).

The following energy conditions are imposed: the weak, strong and dominant energy conditions.

Weak energy condition.

Energy density of any matter distribution measured by an observer at any spacetime point must be positive or equal to zero.

Let v^i be four-velocity of an observer; then the weak energy condition can be written as

$$T_{ik} v^i v^k \geq 0, \tag{1}$$

where T_{ik} is the energy-momentum tensor.

The strong energy condition demands that

$$R_{ik} v^i v^k \geq 0, \tag{2}$$

where R_{ik} is the Ricci tensor.

If matter distribution is a perfect fluid, then the condition (2) gives

$$\rho + P \geq 0. \tag{3}$$

where ρ and P are energy density and pressure, respectively.

Dominant energy condition.

Any matter distribution must move along non-spacelike worldlines. For any non-spacelike vector v^i this condition gives

$$-T_k^i v^k = tlnvf, \tag{4}$$

where $tlnvf$ stands for the timelike or null vector field. It means that this vector must be a non-spacelike one. The condition (4) gives

$$\rho \geq P. \tag{5}$$

For most known matter distributions, all three conditions are valid. However, there are some models where one of the conditions is violated. For example, the strong energy condition is violated inside a regular black hole (Dymnikova 1992; 2002).

The strength of the central singularity

We must understand that singularities which we consider must be genuine and cannot be removed by proper coordinate transformation. If a singularity can be removed, then it is weak; if it cannot be removed, then it is a strong one. A singularity is called gravitationally strong or simply strong if it destroys any object that falls into it by stretching or crushing. If not, such a singularity is called gravitationally weak (Nolan 1999).

According to the Tipler's definition (Tipler 1977), a singularity is gravitationally strong if

$$\lim_{\tau \rightarrow 0} \tau^2 R_{ik} k^i k^k \geq 0, \quad (6)$$

where τ is an affine parameter and k^i is a tangent vector to a non-spacelike geodesic.

The apparent horizon

An apparent horizon is a marginally trapped surface (Faraoni 2015) and a border of the trapped region. To calculate the apparent horizon (Poisson 2007), it is necessary to take the geodesic congruence and the vector u^i which is tangent to this congruence

$$u^i_{;k} u^k = \alpha u^i, \quad (7)$$

where ';' is a covariant derivative, and α shows the failure of u^i to be parallel transported along the congruence. If $\alpha = 0$, the u^i vector is parallel transported along the geodesic.

First of all, the expansion θ should be defined. There are two cases:

$$\theta = u^i_{;i}, \quad (8)$$

if $\alpha = 0$, and

$$\theta = e^\gamma (u^i_{;i} - \alpha), \quad (9)$$

if $\alpha \neq 0$. Here γ is a coefficient which does not have any impact on the apparent horizon definition. A surface is called a trapped one if $\theta < 0$ everywhere on this surface. The apparent horizon, as said earlier, is a border of the trapped region and can be defined as

$$\theta = 0. \quad (10)$$

It follows from (10) that the parameter (9) does not have any impact on the apparent horizon definition.

The general case

The most general spherically symmetric metric in comoving coordinates $\{t, \kappa, \theta, \varphi\}$ has the following form (Joshi 2007):

$$\begin{aligned} ds^2 &= -e^{2\mu} dt^2 + e^{2\psi} dr^2 + R^2 d\Omega^2, \\ d\Omega^2 &= d\theta^2 + \sin^2\theta d\varphi^2. \end{aligned} \quad (11)$$

Here

$$\begin{aligned} \mu &= \mu(r, t), \\ \psi &= \psi(r, t), \\ R &= R(r, t). \end{aligned} \quad (12)$$

The EMT in the case of a perfect fluid is determined by

$$T_{ik} = (\rho + P)u_i u_k + P g_{ik} , \quad (13)$$

where P and ρ are pressure and density, respectively, and u_i is four-velocity.

In any comoving frame the EMT for the first type of the matter field (i.e. for any physically relevant matter such as dust, perfect fluid, scalar fields, etc.) has the form

$$\begin{aligned} T^{00} &= -\rho(r, t) , \\ T^{11} &= P_r(r, t) , \\ T^{22} &= T^{33} = P_\theta(r, t) , \end{aligned} \quad (14)$$

where P_r and P_θ are radial and tangent pressure components, respectively.

If there is a family of non-spacelike future-directed geodesics which terminate at the central singularity in the past, and if the time of the singularity formation is less than the time of the apparent horizon formation, then the result of the collapse is a naked singularity. According to Joshi (2007), the result of such a collapse depends on the initial profile data.

The equation of the apparent horizon in case of a general spherically symmetric metric (11) is

$$g^{ik} R_{,i} R_{,k} = 0 . \quad (15)$$

A radial null geodesic is defined as

$$\frac{dt}{dr} = e^{2(\psi-\mu)} . \quad (16)$$

If the equation (15) cannot be satisfied, then the apparent horizon is absent. And if the condition

$$x_0 = \lim_{t \rightarrow 0, r \rightarrow 0} \frac{dt}{dr} \geq 0 \quad (17)$$

is satisfied and the value x_0 is finite, then the singularity at $r = 0$ and at the time $t = 0$ is a naked one.

In this case there is a family of non-spacelike future directed geodesics which terminate at the central singularity in the past. It means that an external observer can observe processes near this region with extremely high energy density.

The temporary naked singularity in generalised Vaidya spacetime

This paper considers gravitational collapse of thin radiating shells (Vertogradov 2016). The first shell collapses at the central singularity at $r = v = 0$ where $M(0, 0) = 0$. During the collapse of other shells the mass function is growing, and when the last shell collapses, the mass function becomes a well-known Schwarzschild mass. This article only considers shell-focusing singularities, while shell-crossing singularities are gravitationally weak and irrelevant to this study. If there is a family of non-spacelike future-directed geodesics which originate at the central singularity in the past, and if the time of the singularity formation is less than the time of the apparent horizon formation, then the result of such a gravitational collapse is a naked singularity. If there is no such family of geodesics, or the time of the apparent horizon formation is less than the time of the singularity formation, then the result is a black hole.

Generalised Vaidya spacetime corresponds to the combination of two matter fields of types I and II (where type I is the usual matter and type II goes along null geodesics) which is generally described by the equation (Wang, Yu 1999)

$$\begin{aligned} ds^2 &= -e^{2\psi(v,r)} \left(1 - \frac{2M(v,r)}{r} \right) dv^2 + 2\varepsilon e^{\psi(v,r)} dv dr + r^2 d\omega^2 , \\ d\omega^2 &= d\theta^2 + \sin^2(\theta) d\varphi^2 , \end{aligned} \quad (18)$$

where $M(v, r)$ is the mass function depending on coordinates r and v which correspond to advanced/retarded time, means $\varepsilon = \pm 1$ ingoing/outgoing radiating thin shells, respectively.

If we consider gravitational collapse, then $\varepsilon = +1$. With a suitable choice of coordinates, it is possible to set $\psi(v, r) = 0$. So, equation (18) now can be written as

$$ds^2 = - \left(1 - \frac{2M(v, r)}{r} \right) dv^2 + 2\varepsilon dvdr + r^2 d\omega^2 . \quad (19)$$

The energy momentum tensor can be written as

$$T_{ik} = T_{ik}^{(n)} + T_{ik}^{(m)} , \quad (20)$$

where the first term corresponds to the matter field of type I, and the other one corresponds to the matter field type II.

The expression of the energy momentum tensor can be expressed as follows:

$$\begin{aligned} T_{ik}^{(n)} &= \mu L_i L_k , \\ T_{ik}^{(m)} &= (\rho + P)(L_i N_k + L_k N_i) + P g_{ik} , \\ \mu &= \frac{2\dot{M}}{r^2} , \\ \rho &= \frac{2M'}{r^2} , \\ P &= -\frac{M''}{r} , \\ L_i &= \delta_i^0 , \\ N_i &= \frac{1}{2} \left(1 - \frac{2M}{r} \right) \delta_i^0 - \delta_i^1 , \\ L_i L^i &= N_i N^i = 0 , \\ L_i N^i &= -1 , \end{aligned} \quad (21)$$

where P means pressure, ρ means density and L, N mean two null vectors (Wang, Yu 1999).

Strong and weak energy conditions demand the following:

$$\begin{aligned} \mu &\geq 0 , \\ \rho &\geq 0 , \\ P &\geq 0 . \end{aligned} \quad (22)$$

The dominant energy condition imposes the following conditions on the energy momentum tensor:

$$\begin{aligned} \mu &\geq 0 , \\ \rho &\geq P \geq 0 . \end{aligned} \quad (23)$$

Let us consider the simplest case, when the equation of the state is

$$P = \frac{\rho}{3} . \quad (24)$$

(The results for the equation of the state $P = \beta\rho$ where β belongs to the interval $(0, \frac{1}{3})$ are the same (Vertogradov 2016).

If the equation (21) is applied, the mass function is obtained:

$$\begin{aligned} 3M''(v, r)r + 2M'(v, r) &= 0 , \\ M(v, r) &= C(v) + D(v)r^{\frac{1}{3}} . \end{aligned} \quad (25)$$

Pressure and density are defined as

$$\begin{aligned} \rho &= \frac{2}{3} \frac{D(v)}{r^{\frac{8}{3}}}, \\ P &= \frac{2}{9} \frac{D(v)}{r^{\frac{8}{3}}}. \end{aligned} \tag{26}$$

Strong, weak and dominant energy conditions demand that

$$\begin{aligned} D(v) &\geq 0, \\ \dot{C}(v) + \dot{D}(v)r^{\frac{1}{3}} &\geq 0, \\ C(v) &\geq 0. \end{aligned} \tag{27}$$

Moreover, the condition $M(0, 0) = 0$ demands that

$$C(0) = 0. \tag{28}$$

The equation of the apparent horizon is

$$r = 2C(v) + 2D(v)r^{\frac{1}{3}}. \tag{29}$$

Thus, when t is the time of the singularity formation, the equation of the apparent horizon is

$$r = 2D(0)r^{\frac{1}{3}}. \tag{30}$$

If $D(0) > 0$, then $r > 0$ and the time of the apparent horizon formation is less than the time of the singularity formation; in this case, a black hole is a result of gravitational collapse. Thus, naked singularity is only formed when $D(0) = 0$.

Let us consider if there exists a family of non-spacelike future-directed geodesics which originate at the central singularity $r = v = 0$ in the past. Firstly, it is necessary to substitute the mass function (25) into the geodesic equation

$$\frac{dv}{dr} = \frac{2r}{r - 2(C(v) + D(v)r^{\frac{1}{3}})}. \tag{31}$$

If the limit is considered at $r \rightarrow 0, v \rightarrow 0$ in (31), then conditions for the mass function, necessary for the gravitational collapse to result in a naked singularity, are as follows:

$$\begin{aligned} D(0) &= 0, \\ \lim_{v \rightarrow 0, r \rightarrow 0} \frac{C(v)}{r} &= a \geq 0, \\ \lim_{v \rightarrow 0, r \rightarrow 0} \frac{D(v)}{\frac{2}{r^{\frac{1}{3}}}} &= b \geq 0, \\ 2a + 2b &< 1, \end{aligned} \tag{32}$$

where a, b are arbitrary constants.

Thus, when the conditions (32) are satisfied, the collapse results in a naked singularity. Let us consider an explicit example.

Let

$$\begin{aligned} C(v) &= \alpha v, \alpha \geq 0, \\ D(v) &= \beta v, \beta \geq 0. \end{aligned} \tag{33}$$

The condition $D(0) = 0$ is satisfied; therefore, it is necessary to consider whether a family of non-spacelike future-directed geodesics exists. For this purpose, (33) is substituted into (31):

$$\frac{dv}{dr} = \frac{2r}{r - 2\alpha v - 2\beta vr^{\frac{1}{3}}}. \quad (34)$$

Now let us use the definition $x_0 = \lim_{r \rightarrow 0, v \rightarrow 0} \frac{dv}{dr}$ in (34):

$$\begin{aligned} x_0 &= \frac{2}{1 - 2\alpha x_0}, \\ 2\alpha x_0^2 - x_0 + 2 &= 0. \end{aligned} \quad (35)$$

It should be noted that for a family of non-spacelike future-directed geodesics to exist, it is necessary and sufficient that one root of the equation (35) should be positive. It is possible if

$$\alpha < \frac{1}{8}. \quad (36)$$

If the condition (36) is satisfied, then the singularity is naked.

The negative pressure case in generalised Vaidya spacetime

In this case the equation of the state $P = -\frac{1}{2}\rho$ is considered. Then the mass function is given by

$$M(v, r) = c(v) + d(v)r^2. \quad (37)$$

In this model the strong energy condition is violated, because $P < 0$. Weak and dominant energy conditions demand that

$$\dot{c}(v) + \dot{d}(v)r^2 \geq 0, d(v) \geq 0. \quad (38)$$

Since $M(0, 0) = 0$, it follows that $c(0) = 0$.

Substituting this solution into the equation of the apparent horizon, we find

$$\begin{aligned} 1 - \frac{2M(v, r)}{r} &= 0, \\ 1 - 2d(v)r - \frac{2c(v)}{r} &= 0. \end{aligned} \quad (39)$$

When $c(0) = 0$ and $d(0) = 0$, the apparent horizon is absent. Solving this equation with respect to r gives

$$\begin{aligned} 2d(v)r^2 - r + 2c(v) &= 0, \\ D &= 1 - 16c(v)d(v). \end{aligned} \quad (40)$$

It follows that the apparent horizon is absent when $(v)d(v) > \frac{1}{16}$. Thus, the naked singularity formation is present when $v = 0$, when the apparent horizon appears, and later when $c(v)d(v) > \frac{1}{16}$; then the apparent horizon disappears again.

The next question under consideration is the existence of non-spacelike future-directed geodesics which terminate at the singularity in the past. For example, let us consider the radial null geodesics equation

$$\frac{dv}{dr} = \frac{2r}{r - 2c(v) - 2d(v)r^2}. \quad (41)$$

Let us denote $\lim_{r \rightarrow 0, v \rightarrow 0} \frac{dv}{dr} = x_0$, then we can substitute $d(v) = \xi v$ and $c(v) = \lambda v$ (where λ, ξ are real positive constants) in the geodesics equation and rewrite it as

$$\begin{aligned} x_0 &= \frac{2}{1 - 2\lambda x_0}, \\ 2\lambda x_0^2 - x_0 + 2 &= 0, \\ x_{0\pm} &= \frac{1 \pm \sqrt{1 - 16\lambda}}{4}. \end{aligned} \quad (42)$$

The family of non-spacelike future-directed geodesics exists if there exists a positive root for the equation above. x_{0+} is positive, but λ should be limited as

$$1 - 16\lambda \geq 0 \rightarrow \lambda \leq \frac{1}{16}. \quad (43)$$

The result of such a collapse might be a naked singularity.

At the late stage of v , expression (40) has two positive roots. Thus, there are three regions:

- region I, where $\theta < 0 \rightarrow 0 \leq r < r_1$,
- region II, where $\theta > 0 \rightarrow r_1 < r < r_2$,
- region III, where $\theta < 0 \rightarrow r_2 < r < +\infty$.

Let us consider the expansion θ :

$$\theta = e^{-\gamma} \frac{2}{r} \left(1 - \frac{2c(v) + 2d(v)r^2}{r} \right) \quad (44)$$

where $e^{-\gamma}$ has no impact on the sign of θ .

As it follows from (44):

$$\begin{aligned} \theta < 0 &\rightarrow 0 \leq r < r_1, \\ \theta > 0 &\rightarrow r_1 < r < r_2, \\ \theta < 0 &\rightarrow r_2 < r < +\infty. \end{aligned} \quad (45)$$

This result is similar to Schwarzschild-de Sitter-Kottler spacetime (Faraoni 2015), but with the equation of the state $p = -\frac{1}{2}\rho$. It is clear that region I is a black hole solution with a singularity in its center. If an observer is located in region II, then he can not get information from regions I and III.

In this case the result of the gravitational collapse might be a naked singularity; however, it is temporary and in a very short period of time it is covered with the apparent horizon. However, this is a white hole apparent horizon, because it follows from (44) that $\theta > 0$ at $0 < r < r_2$. In this case, $r_1 < 0$. Thus, there is a region of spacetime where geodesics cannot get into region $0 < r < r_2$ from region $r_2 < r < +\infty$. This white hole is shrinking. At the late stage of v the expression (44) has two positive roots. Then two apparent horizons merge, and the collapse results in a singularity, which is similar to the singularity of the future in Friedman model.

A naked singularity is eternal if there is a family of non-spacelike future-directed geodesics which terminate at the central singularity in the past, the apparent horizon never appears and θ is positive everywhere.

Let us consider the case of the negative pressure when θ is always positive. The equation of the state in this case is $P = -\alpha\rho$, where $1 \leq \alpha < 0$. cannot be more than one due to energy conditions which demand $\rho \geq P \geq 0$. The mass function in this case is defined as

$$M(v, r) = c(v) + d(v)r^{1+2\alpha}. \quad (46)$$

The expansion in this case is given by

$$\theta = e^{-\gamma} \frac{2}{r} \left(1 - \frac{2c(v) + 2d(v)r^{1+2\alpha}}{r} \right). \quad (47)$$

Here $e^{-\gamma}$ has no impact on the sign of θ .
Then θ is restricted to positive numbers:

$$\theta > 0 \rightarrow r > 2c(v) + 2d(v)r^{1+2\alpha}. \quad (48)$$

It follows from (48) that this condition can only be satisfied if $c(v) \equiv 0$. Now α is restricted to $\alpha < 1$, because if $\alpha = 1$, then there is no singularity. The final equation is

$$\begin{aligned} \theta > 0 &\rightarrow 1 - 2d(v)r^{2\alpha} > 0, \\ r^{2\alpha}d(v) &< \frac{1}{2}. \end{aligned} \quad (49)$$

It also follows from (49) that there is no apparent horizon, because the condition to the apparent horizon formation is $\theta = 0$, but θ is restricted to $\theta > 0$. Moreover, (49) shows that $d(v)$ must be very small and, therefore, the pressure must be very small, too, because

$$P = -4\alpha(1 + 2\alpha) \frac{d(v)}{r^{2-2\alpha}}$$

In order to prove that there is a family of non-spacelike future-directed geodesics which terminate at the central singularity in the past, it is necessary to consider a null radial geodesic which is given by

$$\frac{dv}{dr} = \frac{2}{1 - 2d(v)r^{2\alpha}}. \quad (50)$$

This family exists if

$$\lim_{v \rightarrow 0, r \rightarrow 0} \frac{dv}{dr} \geq 0. \quad (51)$$

However, $\frac{dv}{dr}$ is always positive, because $1 - 2d(v)r^{2\alpha}$ must be positive due to the condition $\theta > 0$.

Thus, all three conditions are satisfied and, according to them, the result of such a collapse is an eternal naked singularity.

Let us look at the equation (49). It has a solution

$$r_{aah} = \left(\frac{1}{2d(v)} \right)^{\frac{1}{1+2\alpha}}. \quad (52)$$

This solution must be matched to the exterior Schwarzschild one. Let r_m be a boundary of two spacetimes. Then there are two cases:

1. $r_{aah} < r_m$,
2. $r_{aah} > r_m$.

In the first case the result of the gravitational collapse is a white hole, because θ is positive at $0 \leq r < r_{ah}$, and there is a family of non-spacelike future-directed geodesics which terminate at the central singularity in the past. However, $\frac{dv}{dr}$ is always positive inside the white hole apparent horizon, and there is no geodesic which can get into the white hole from the region outside of the apparent horizon. But in this case it takes endless time to get to the exterior observer from the singularity.

In case II solutions are matched at r_m , which is less than r_{ah} , and it takes a finite time to get to the exterior observer from the central singularity. Now let us look at the expression of pressure which is given by

$$P = -2\alpha(1 + 2\alpha) \frac{d(v)}{r^{1+2\alpha}}. \quad (53)$$

It follows from equations (49) and (53) that the less the pressure is, the more r_{ah} is. It means that an eternal naked singularity can be formed if the pressure is small enough.

This paper does not consider the question about the strength of the central singularity, but all considered singularities are gravitationally strong (see, for example, Vertogradov 2016; 2018).

Conclusion

This paper briefly explained the naked singularity phenomenon. It identified conditions when a naked singularity might be formed in the case of the most general spherically symmetric metric. Moreover, it presented two explicit examples of the naked singularity formation. This paper considered only spherically symmetric objects. There has been no analytical model for the gravitational collapse of a rotating object proposed so far.

References

- Dymnikova, I. (2002) The cosmological term as a source of mass. *Classical and Quantum Gravity*, 19 (4), 725–739. DOI: 10.1088/0264-9381/19/4/306 (In English)
- Dymnikova, I. (1992) Vacuum nonsingular black hole. *General relativity and gravitation*, 24 (3), 235–242. DOI: 10.1007/BF00760226 (In English)
- Faraoni, V. (2015) *Cosmological and black hole apparent horizons*. Cham: Springer, 199 p. DOI: 10.1007/978-3-319-19240-6 (In English)
- Goswami, R., Joshi, P. S. (2007) Gravitational collapse of a self-interacting scalar field. *Modern Physics Letters A*, 22 (01), 65–74. DOI: 10.1142/S0217732307020701 (In English)
- Hawking, S. W., Ellis, G. F. R. (1973) *The large scale structure of space-time*. Cambridge: Cambridge University Press, 391 p. DOI: 10.1017/CBO9780511524646 (In English)
- Joshi, P. S. (2007) *Gravitational collapse and spacetime singularities*. Cambridge: Cambridge University Press, 273 p. DOI: 10.1017/CBO9780511536274 (In English)
- Mkenyeleye, M. D., Goswami, R., Maharaj, S. D. (2014) Gravitational collapse of generalized Vaidya spacetime. *Physical Review D*, 90 (6), 064034. DOI: 10.1103/PhysRevD.90.064034 (In English)
- Nolan, B. C. (1999) Strengths of singularities in spherical symmetry. *Physical Review D*, 60 (2), 024014. DOI: 10.1103/PhysRevD.60.024014 (In English)
- Oppenheimer, J. R., Snyder, H. (1939) On continued gravitational contraction. *Physical Review*, 56 (5), 455–459. DOI: 10.1103/PhysRev.56.455 (In English)
- Poisson, E. (2004) *A relativist's toolkit: The mathematics of black-hole mechanics*. Cambridge, UK; New York: Cambridge University Press, 233 p. DOI: 10.1017/CBO9780511606601 (In English)
- Tipler, F. J. (1977) Singularities in conformally flat spacetimes. *Physics Letters A*, 64 (1), 8–10. DOI: 10.1016/0375-9601(77)90508-4 (In English)
- Vertogradov, V. D. (2016) Naked singularity formation in generalized Vaidya space-time. *Gravitation and Cosmology*, 22 (2), 220–223. DOI: 10.1134/S020228931602016X (In English)
- Vertogradov, V. (2018) The eternal naked singularity formation in the case of gravitational collapse of generalized Vaidya space-time. *International Journal of Modern Physics A*, 33 (17), 1850102. DOI: 10.1142/S0217751X18501026 (In English)
- Wang, A., Wu, Y. (1999) Generalized vaidya solutions. *General Relativity and Gravitation*, 31 (1), 107–114. DOI: 10.1023/A:1018819521971 (In English)