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Analytical regularities of inversionless superradiance

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Abstract. The article reports the results of a theoretical study of superradiance of three-level optical systems with a doublet in the ground state (Λ-scheme) placed in a high-quality cavity. Hyperbolic chaos and unpredictable dynamic movements of the system appear on the surface of a multidimensional torus without dissipative losses in the conditions of superradiance without population inversion. The study resulted in the development of conservation laws to reduce the dimension of the phase space. An analytical result is obtained for the special case of a degenerate doublet.

Keywords: superradiance, superradiance without inversion, Λ-scheme, the Duffing equation

Introduction

It is well-known that the necessary condition for Dicke superradiance (SR) (Dicke 1954) is the presence of the initial population inversion of transitions levels (Andreev et al. 1980; 1993; Benedict et al. 1996; Bonifacio et al. 1971; Bonifacio, Lugiato 1975; Gross, Haroche 1982; Kalachev, Samartsev 2003; MacGillivray, Feld 1976; 1981; Rehler, Eberly 1971; Sokolov, Trifonov 1974; Zheleznyakov et al. 1989). Considering our case with three-level Λ-emitters, this restriction is not mandatory. Superradiance is possible even when the initial upper-level population is less than the total population of the lower doublet—SR without population inversion (Carlson et al. 1980; Harris 1989; Kocharovskaya 1997; Kocharovskaya, Khanin 1988; Kocharovskaya, Mandel 1990; Malikov, Trifonov 1984; Malyshev et al. 1998; 2000; 2003; Ryzhov et al. 2012; 2017; Scully 1992; Yuan, Svidzinsky 2012; Zaitsev et al. 1999). The essence of the effect is as follows. If one prepares the initial state of the lower doublet as a coherent superposition, transition to which from the upper state is forbidden, then the orthogonal to the initial superposition, transition to which is allowed, appears to be unpopulated. In this case, the transition from the upper level to this superposition state appears to be inverted at an arbitrarily small population of the upper level. The initial coherent state of the doublet can be created by a short low-frequency π/2-pulse (Malyshev et al. 1998; 2003; Ryzhov et al. 2012; 2017; Zaitsev et al. 1999). Crystals activated by rare earth ions, such as LaF_3:Pr^{3+}, Y_2SiO_5:Pr_{3+}, Y_2SiO_5:Eu^{3+}, Y_2SiO_5:Er^{3+}, etc., are real objects where the conditions for observing SR regimes without inversion can be achieved. At cryogenic temperatures, the conditions of the 4f orbitals of these ions are characterized by a high degree of optical coherence, less than kHz, and very low heterogeneous expansion from MHz to GHz (Ryzhov, Vasil'ev, Kosova et al. 2017). An additional level close to the ground one is a solution to the problem in question.

The article focuses on analytical regularities of the nonlinear dynamics of superradiance of an ensemble of three-level Λ-atoms which are spatially homogeneously and isotropically distributed in a high-Q cyclic cavity. The model of SR proposed in this work is conservative. There are no relaxation losses of SR. The time dynamics of the model is considered in terms of the semiclassical approach:

the ensemble of three-level emitters is described by equations for the density matrix ρ_{mn} (m, n = 1, 2, 3), while the electromagnetic field is described by Maxwell's equations. The conservation of the system results in integrals of motion that considerably reduce the dimension of the phase space of the examined model: ($R^{11} \rightarrow R^5$). For the degenerate doublet, we found mapping that reduces the problem of the three-level SR to a nonlinear oscillator with cubic nonlinearity ($R^5 \rightarrow R^2$).

Model and formalism

We consider an ensemble of three-level atoms with the Λ-scheme of operation transitions. The atoms are homogeneously distributed along one of the arms of a high-Q cyclic cavity (Fig. 1).

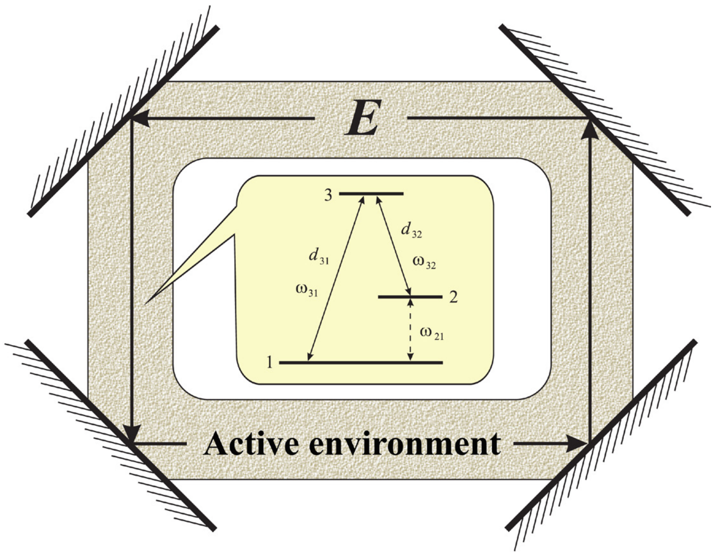

Fig. 1. A diagram of a unidirectional ring cavity. The active medium of Λ-emitters is gray. An inserted picture shows energy-level diagram of Λ-emitters. The number of the level (n = 1,2,3) corresponds to the state of the emitter with energy E_n. Solid and dashed arrows indicate, respectively, the allowed and forbidden transitions between the energy levels of the emitter, with the frequencies of the corresponding transitions ω_{21}, ω_{31}, and ω_{32}, and the transition dipole moments d_{31} and d_{32} ($d_{21} = 0$).

In addition, all vectors (transition dipole moments and polarization of the field) are assumed to be directed identically and perpendicularly to the axis of the system. The evolution of the system then obeys the following (one-dimensional) system of Maxwell—Bloch equations:

$$
\begin{aligned}
\dot{\rho}_{11} &= i\frac{d_{31}E}{\hbar}\left(\rho_{31}-\rho_{13}\right),\\
\dot{\rho}_{22} &= i\frac{d_{32}E}{\hbar}\left(\rho_{32}-\rho_{23}\right),\\
\dot{\rho}_{33} &= -i\frac{d_{31}E}{\hbar}\left(\rho_{31}-\rho_{13}\right)-i\frac{d_{32}E}{\hbar}\left(\rho_{32}-\rho_{23}\right),\\
\dot{\rho}_{21} &= -i\omega_{21}\rho_{21}-i\frac{d_{31}E}{\hbar}\rho_{23}+i\frac{d_{32}E}{\hbar}\rho_{31},\\
\dot{\rho}_{31} &= -i\omega_{31}\rho_{31}-i\frac{d_{31}E}{\hbar}\left(\rho_{33}-\rho_{11}\right)+i\frac{d_{32}E}{\hbar}\rho_{21},\\
\dot{\rho}_{32} &= -i\omega_{32}\rho_{32}-i\frac{d_{32}E}{\hbar}\left(\rho_{33}-\rho_{22}\right)+i\frac{d_{31}E}{\hbar}\rho_{12},\\
&\left(\frac{\partial^2}{\partial x^2}-\frac{1}{c}\frac{\partial^2}{\partial t^2}\right)E=\frac{4\pi}{c^2}\frac{\partial^2 P}{\partial t^2},
\end{aligned}
\tag{1}
$$

where, ρ_{nm} are the elements of the density matrix of the three-level atom at the point with coordinate x at moment of time t (m, n = 123); d_{31} and d_{32} are the dipole moments of the corresponding transitions, which, without loss of generality, can be considered to be real-valued and positive; ω_{31} and ω_{31} are the frequencies of optical transitions between the upper level 3 and the doublet sublevels 1 and 2; ω_{21} is the frequency of the transition between the sublevels of the doublet; $P = N(d_{31}\rho_{31} + d_{32}\rho_{32} + c.c.)$ is the polarization of the medium; N is the concentration of atoms; c is the speed of light in vacuum; E is the electric field strength.

The relaxation of the populations and polarization (homogeneous and related to inhomogeneous broadening) is not taken into account: we assume that the SR time is considerably shorter than all relaxation times and consider the dynamics of superradiance on this scale. In addition, we neglect the decay of the field due to cavity losses. Frequency ω_{21} of the doublet splitting is assumed to be much smaller than frequencies ω_{31} and ω_{32} of the optical transitions. We also assume that the spectrum of SR and the value of doublet splitting ω_{21} do not exceed the spacing between cavity modes, i. e., we restrict ourselves to the single-mode approximation.

We will seek the solution to the system of equations (1) in the form

$$\begin{aligned} \rho_{31} &= R_{31} e^{-i(\omega t - kx)}, \\ \rho_{32} &= R_{32} e^{-i(\omega t - kx)}, \\ E &= A e^{-i(\omega t - kx)} + c.c, \end{aligned} \tag{2}$$

where $k = \omega/c$, while the field amplitude A and off-diagonal elements R_{31} and R_{32} of the density matrix (in what follows, they will be referred to as the high-frequency coherences) are functions that very slowly change on a scale of the optical period $2\pi/\omega$ and the radiation wavelength $\lambda = 2\pi/k$ — the approximation of slowly varying amplitudes (SVA). Note that an analogous assumption with respect to low-frequency coherence ρ_{21} (on a scale of $2\pi/\omega$) is not used. It is natural to assume that passage time L/c (L is the cavity length) is much shorter than characteristic times of the problem, i. e., during one round trip of the light in the cavity, the state of the medium changes insignificantly. In this case, the retardation can be neglected. Then the field at the input into the active medium (by virtue of a high-quality factor of the cavity) is equal to the field at its output, which also justifies the use of the mean-field approximation. And, finally, let us assume that the $|3\rangle \leftrightarrow |1\rangle$ and $|3\rangle \leftrightarrow |2\rangle$ transition dipole moments are identical ($d_{31} = d_{32} = d$). This is the approximation that is not principal for the problem under consideration.

Taking a standard path from the system of equations (1) to a similar system for SVA, we obtain

$$\dot{\rho}_{11} = ER_{31}^* + E^* R_{31}, \quad \dot{\rho}_{22} = ER_{32}^* + E^* R_{32}, \tag{3a}$$

$$\dot{\rho}_{33} = -\left(ER_{31}^* + E^* R_{31}\right) - \left(ER_{32}^* + E^* R_{32}\right), \tag{3b}$$

$$\dot{\rho}_{21} = -i\delta\dot{\rho}_{21} + ER_{32}^* + E^* R_{31}, \tag{3c}$$

$$\dot{R}_{31} = -\frac{i\delta}{2} R_{31} + E\left(\rho_{33} - \rho_{11} - \rho_{21}\right), \tag{3d}$$

$$\dot{R}_{32} = \frac{i\delta}{2} R_{32} + E\left(\rho_{33} - \rho_{22} - \rho_{21}^*\right), \tag{3e}$$

$$\dot{E} = R_{31} + R_{32} \tag{3f}$$

here, dots denote the derivatives with respect to the dimensionless time $\tau = t\Omega$, where $\Omega^{-1} = \sqrt{\hbar\left(2\pi\omega d^2 N\right)^{-1}}$ is the constant that determines the time scale (Ω^{-1}); $\delta = \omega^{21}/\Omega$ is the dimensionless splitting frequency of the doublet; and $E = -idA/(\hbar\Omega)$ is the dimensionless amplitude of the electric field strength. For simplicity, eigenfrequency $\omega = (\omega_{31} + \omega_{32})/2$ of the cavity is considered to be centered between frequencies ω_{31} and ω_{32}.

This system of equations has the following integrals of motion:

$$|E|^2 + \rho_{33} = \text{const}, \tag{4a}$$

$$\rho_{11} - \rho_{22} - \rho_{33} = 1, \tag{4b}$$

$$\rho_{11}^2 + \rho_{22}^2 + \rho_{33}^2 + 2\left(\left|\rho_{21}\right|^2 + \left|R_{31}\right|^2 + \left|R_{32}\right|^2\right) = \text{const}, \tag{4c}$$

(4a) shows the law of conservation of the excitation energy of the system, (4b) and (4c) represent the normalization conditions for the density matrix and its square respectively. The presence of integrals of motion makes it possible to considerably simplify the analysis of the dynamics of the three-level SR.

Initial conditions, symmetry, and simplification of the model

The presence of a doublet in the ground state introduces new effects into the response of the system. They are generated by the competition between the transitions $|3\rangle \leftrightarrow |1\rangle$. In connection with this, to investigate the kinetics of the three-level SR, we will choose such initial conditions that will ensure most effective interaction between the parts of the system "cavity + atoms + field", i. e., at any initial population of the upper state and with a minimal delay of the SR pulse. In this regard, let us focus on equations (3d) and (3e) for the high-frequency coherences R_{31} and R_{32}. They contain terms that are proportional to low-frequency coherence ρ_{21}. In this case, if ρ_{21} (0) ≠ 0, the evolution of initial fluctuations of R_{31} will depend on phase ρ_{21} (0). At positive values of ρ_{21} (0) these fluctuations will decrease; however, if the values of ρ_{21} (0) are negative, these fluctuations, on the contrary, will increase avalanche-like, leading to superradiance. Notably, this possibility arises at any difference of the populations in channels 3 ↔ 1 and 3 ↔ 2 due to the transformation of the low-frequency coherence ρ_{21} (0) into high-frequency coherences R_{31} and R_{32}. The latter effect is explicitly reflected in the integral of motion (4c). The analysis of superradiance of this Λ-system is significantly simplified upon passage to a new basis $|3\rangle$, $|+\rangle = \left(|1\rangle + |2\rangle\right)/\sqrt{2}$, $|-\rangle = \left(|1\rangle - |2\rangle\right)/\sqrt{2}$, (Malyshev et al. 1998; Ryzhov et al. 2012; 2017; Zaitsev et al. 1999). In this case, the elements of the density matrix are transformed in accordance with the following relations:

$$\begin{gathered}
\rho_{++} = \frac{1}{2}(\rho_{11} + \rho_{22} + 2Re\left[\rho_{21}\right], \\
\rho_{--} = \frac{1}{2}(\rho_{11} + \rho_{22} - 2Re\left[\rho_{21}\right], \\
\rho_{+-} = \frac{1}{2}\left(\rho_{11} - \rho_{22} + \rho_{21} - \rho_{21}^*\right), \\
R_{3+} = \frac{1}{\sqrt{2}}\left(R_{31} + R_{32}\right), \quad R_{3-} = \frac{1}{\sqrt{2}}\left(R_{31} - R_{32}\right),
\end{gathered} \tag{5}$$

where ρ_{++} and ρ_{--} are the populations of the active and passive states respectively; ρ_{++} is the low-frequency coherence; and R_{32} and R_{32} are the high-frequency coherences of the corresponding optical channels.

It can be seen from the expression for population ρ_{++} of the active state presented in the relations (5) that for the three-level SR to take place, the presence of an inversion population in active channel $|3\rangle \leftrightarrow |+\rangle$ is necessary, i. e., at the initial moment of time, inequality ρ_{33} (0) > ρ_{++} (0) should be implement. In the ideal case, in which the population of the active state is zero, $\rho_{++}(0) = \rho_{11}(0) + \rho_{22}(0) + 2\mathrm{Re}[\rho^{21}(0)] = 0$, the following conditions should be met:

$$\begin{gathered}
Re\left[\rho_{21}(0)\right] = ((\alpha - 1))/2, \quad \mathrm{Im}\left[\rho_{21}(0)\right] = 0, \\
-Re\left[\rho_{21}(0)\right] = \rho_{11}(0) + \rho_{22}(0),
\end{gathered} \tag{6}$$

where ρ_{22} (0) = α and 0 < α ≤1. In what follows, we will assume that the lower doublet is prepared in a maximally coherent state if conditions (6) are met at the initial moment of time. We emphasize again that, under these starting conditions, superradiance can occur at any initial population ρ_{33} (0) of the upper state, even if there is no inversion population on the whole, when the total initial population of the lower doublet exceeds the initial population of the upper level, ρ_{11} (0) + ρ_{22} (0) > ρ_{33} (0).

If the initial electric field strength is zero,

$$Re\left[E\left(0\right)\right]=Im\left[E\left(0\right)\right]=0 \tag{7}$$

for superradiance to arise, it suffices to set small seed values of the high-frequency coherences, e. g.:

$$Re\left[R_{31}\left(0\right)\right]=Re\left[R_{32}\left(0\right)\right]=\pm R_{0}, \tag{8}$$

where, without loss of generality, it is assumed that Im $[R_{31}(0)]$ = Im $[R_{32}(0)] = 0$, while the value of $R_0 \ll 1$. We are not interested in the fluctuations of SR; therefore, initial values $R_{31}(0)$ and $R_{32}(0)$ are specified as determinate parameters, which corresponds to the conditions of the induced SR (Carlson et al. 1980; Malikov, Trifonov 1984).

The system of differential equations (3) with initial conditions (6)–(8) was solved numerically. The following two controlling parameters varied: the initial population $\rho_{33}(0) = \alpha$ of the upper level and the splitting frequency δ of the doublet. This returned a number of interesting regularities of the time dynamics of the amplitudes of the electric field of SR and elements of the density matrix: $Re[E(\tau)] \neq 0$, $Im[E(\tau)] = 0$; the real parts of high-frequency coherences $Re[R_{31}(\tau)]$ and $Re[R_{32}(\tau)]$ show a similar behavior, whereas their imaginary parts $Im[R_{31}(\tau)]$ and $Im[R_{32}(\tau)]$ exhibit an antiphase behavior. In accordance with this, the squares of their moduli, $|R_{31}|^2 = |R_{32}|^2$, evolve identically. The dynamics of populations $\rho_{11}(\tau)$ and $\rho_{22}(\tau)$ are identical and repeat the dynamics of superradiance field intensity $|E|^2$ (Ryzhov et al. 2017). This makes it possible to considerably simplify the mathematical model of the problem under consideration.

By introducing the notation

$$Re\left[E\right]=\epsilon,\quad Im\left[E\right]=0,\quad \rho_{21}=\eta+i\chi, \tag{9a}$$

$$\rho_{11}=\rho_{22}=\left(1-\rho_{33}\right)/2,\qquad \rho_{33}=\alpha-\epsilon^{2}, \tag{9b}$$

$$\begin{aligned} &Re\left[R_{31}\right]=Re\left[R_{32}\right]=\xi,\\ &Im\left[R_{31}\right]=-Im\left[R_{32}\right]=\zeta,\\ &\left|R_{31}\right|^{2}=\left|R_{32}\right|^{2}=\xi^{2}+\zeta^{2}, \end{aligned} \tag{9c}$$

we can transform the system of differential equations (3) into the following system:

$$\dot{\epsilon}=2\xi, \tag{10a}$$

$$\dot{\xi}=-\frac{\delta}{2}\zeta+\frac{1}{2}\left(3\alpha-1\right)\epsilon-\frac{3}{2}\epsilon^{3}-\epsilon\eta, \tag{10b}$$

$$\dot{\zeta}=-\frac{\delta}{2}\xi-\epsilon\chi, \tag{10c}$$

$$\dot{\eta}=\delta\chi+2\epsilon\xi, \tag{10d}$$

$$\dot{\chi}=-\delta\eta+2\epsilon\zeta. \tag{10e}$$

Therefore, the relations (9) implement the reduction of our model from the complex domain to the real one. As a consequence, initial phase space R^{11} (3) of the model is completely mapped into R^5 (10). In addition, taking into account (4b) and relations (9), integral of motion (4c) takes the form

$$\begin{aligned} &(\epsilon^{2}-\gamma)^{2}+\frac{4}{3}\left(\eta^{2}+\chi^{2}+2\xi^{2}+2\zeta^{2}\right)=const,\\ &const=\frac{4}{3}\left(\alpha^{2}-\gamma^{2}+2R_{0}^{2}\right),\quad \gamma=\alpha-\frac{1}{3}. \end{aligned} \tag{11}$$

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It is important to note that this law of conservation restricts the domain of existence of phase trajectories of the system and determines a closed hypersurface in the phase space (ϵ, ξ, ζ, η and χ) outside of which solutions of the system of equations (10) do not exist at any values of parameters α and δ. This makes it possible to characterise the process of SR as a process that is stable in the sense of Lagrange (Kuznetsov 2001). Topological specific features of hypersurface (11) depend on the sign of constant γ. First, $1 \geq \alpha > \frac{1}{3}$, i. e., at γ > 0, this is a five-dimensional "dumbbell" with symmetry axis $\boldsymbol{\epsilon}$. Second, if i. e., γ > 0, the hyper-surface is a five-dimensional ellipsoid. In the first variant, in the phase space of the system (under the condition α 1/3), there is hyperbolic chaos related with unpredictable abrupt transitions of the representing points between the family of torus lying in different cavities of "dumbbells". The movement of the phase space takes place on the surface of those tori. In the second case (α → 0), hyperbolic chaos is also present, but the family of tori already lying in the elliptical phase space intersects themselves in many ways, creating conditions that Puankare defined as a homoclinic structure (Puankare 1972; Ryzhov et al. 2017) of dynamic chaos.

Degenerate doublet

Let us consider a particular case of a degenerate doublet (δ = 0). In this limit, the system of differential equations (10) is considerably simplified and takes the form

$$\dot{\epsilon} = 2\xi, \quad \dot{\eta} = 2\epsilon\xi, \tag{12a}$$

$$\dot{\chi} = 2\epsilon\zeta, \quad \zeta = -\epsilon\chi, \tag{12b}$$

$$\dot{\xi} = \frac{1}{2}(3\alpha - 1)\epsilon - \frac{3}{2}\epsilon^3 - \epsilon\eta. \tag{12c}$$

This system of equations has integrals of motion. First, Eq. (12a) along with the initial conditions ϵ (0) = 0 and η (0) = – (1–α)/2 yield the first integral of motion:

$$2\eta - \epsilon^2 = \alpha - 1. \tag{13}$$

Second, Eq. (12b) and the initial conditions χ(0) = ζ(0) = 0 yield the second integral of motion: $\chi^2 + 2\zeta^2 = 0$, which means that functions χ(τ) and ζ(τ), which are defined in the real domain, remain unchanged and equal to zero within the entire SR process: χ(τ) = ζ(τ) = 0, τ ≥ 0. Then, expressing functional dependence η(ϵ) from (13) via ϵ^2 and substituting it into (12c), we obtain

$$\dot{\epsilon} = 2\xi, \tag{14a}$$

$$\dot{\xi} = \alpha\epsilon - 2\epsilon. \tag{14b}$$

Eliminating variable ξ from (14), we arrive at the following closed equation for the field ϵ (15)

$$\ddot{\epsilon} - 2\alpha\epsilon + 4\epsilon^3 = 0, \quad 1 \geq \alpha > 0, \tag{15}$$

which represents the Duffing equation (Duffing 1918) for an oscillator with a cubic nonlinearity without friction and external driving force. Eq. (15) yields the third integral of motion:

$$\frac{1}{2}\dot{\epsilon}^2 + V(\epsilon) = e, \tag{16}$$

the physical meaning of which is that it corresponds to the total energy of the oscillator (superradiance field), where, taking into account the initial conditions $\epsilon(0) = 0$ and ξ(0) = ±R_0 ≠ 0, the value of the total energy is $e = 2R_0^2 > 0$ ($R_0 \neq 0$), while the function $V(\epsilon) = \epsilon^4 - \alpha\epsilon^2$ can be interpreted as a potential energy of SR. The relation between the signs in front of the linear and nonlinear terms in (15) characterises the SR field as that of a stable oscillation process in double-humped potential V(ϵ) with infinite walls, which has three singular points in the phase space $(\epsilon, \dot{\epsilon})$: $A_{1,2}\left(\pm\sqrt{\alpha/2}, -\alpha^2/4\right)$ are the points of a stable equilibrium of the centre type (minima of the potential V(ϵ)); and O(0,0) is the point of an unstable equilibrium of the saddle type (maximum of the potential V(ϵ)).

In the general case, ξ(0)=$R_0 \neq 0$, the value of total energy e (16) is always positive, $2R_0^2 > 0$. Consequently, the oscillation process of superradiance is always supernonlinear. In this case, Eq. (16) will have an exact solution. In terms of the elliptic functions, it can be obtained by applying the following substitution:

$$\epsilon = \epsilon_1 cos\left[\varphi\right], \qquad \epsilon_1^2 = \frac{\alpha}{2} + \sqrt{\frac{\alpha}{2} + 2R_0^2}$$

Then,

$$\varphi = am\left[\left(\phi\tau - \tau_d\right);\, m\right], \qquad \epsilon\left(\tau\right) = \epsilon_1 cn\left[\left(\phi\tau - \tau_d\right);\, m\right],$$

$$\tau_d = K\left(m\right), \qquad \phi = \sqrt{2\alpha + \frac{8R_0^2}{\epsilon_1^2}}, \tag{17}$$

$$T = 4K\left(m\right), \qquad m^2 = 1 - \frac{2R_0^2}{4R_0^2 + \alpha\epsilon_1^2},$$

where am[τ'; m] is the amplitude of Jacobi functions; cn[τ'; m]is the elliptic cosine; $\tau' = \phi\tau - \tau_d$, where τ_d is the delay time of the SR pulse (initial stage of SR); *K(m)* is the complete elliptic integral of the first kind; and *T* is the period of oscillations of the electric field strength. In this case, the physical picture of superradiance is rather transparent. From Eq. (14a), we have $\dot{\epsilon}\left(0\right) = \pm 2\xi\left(0\right) = \pm R_0$. If ξ(0) = R_0, then $\dot{\epsilon}\left(\tau\right) = \sqrt{2\left[e - V\left(\epsilon\right)\right]}$ and the field of superradiance will increase ($\dot{\epsilon}\left(\tau\right) > 0$) in the time interval $0 \le \tau < T/4$ with delay τ_d; after that, the field will decrease within the interval of the same length, initiating a superradiance pulse. Upon the reverse motion, the system emits an antiphase pulse, and this process is periodically reproduced, since the system is conservative. If ξ(0) = $-R_0$ and $\dot{\epsilon}\left(\tau\right) = -\sqrt{2\left[e - V\left(\epsilon\right)\right]}$, the field of superradiance has a phase shift by $T/2 = \pi$.

Non-degenerate doublet

Eq. (3) has a numerical solution to study the dynamics of SR. For simplicity, the dipole moments of optical transitions were assumed to be equal: $d_{31} = d_{32} = 1$. Natural frequency of the resonator was chosen as the average between the frequencies of the high-frequency channels $\omega = (\omega_{31} + \omega_{32})/2$.

The calculations were carried out under the following initial conditions: $\rho_{11}(0) = \rho_{22}(0) = 0.4$, $\rho_{33}(0) = 0.2$, $\rho_{21}(0) = -0.4$, $R_{31}(0) = R_{32}(0) = 10^{-8}$, $E(0) = 0$. We can observe that there is no population inversion in the optical channels. At the same time, the inversion between the upper and the active state is 0.2. Non-zero values of $R_{31}(0)$ and $R_{32}(0)$ are required to initiate SR. The periodical regime of SR is observed without splitting doublet condition (see Fig. 2a) described by nonlinear equation (15) with cubic nonlinearity which was previously discussed. The splitting of the lower doublet $\omega_{21} \neq 0$ leads to the appearance of temporal modulation of the SR signals (Figs. 2b, 2c). This is due to the fact that with a nonzero splitting of the lower doublet, the states |+⟩ and |–⟩ are not stationary and, over time, the active state is periodically transformed into a passive one. In addition, a small change in the parameter of doublet splitting δ = 0.05 ↔ δ = 0.1 results in a complete change of the SR generation conditions. The transformation of an active state into a passive one leads to hyperbolic chaos in the system or the chaos of collapsing tori. Figs. 2b, 2c show variants of hyperbolic chaos. The time and amplitude characteristics of finding the system on the surface of one torus can vary unpredictably from those on another torus.

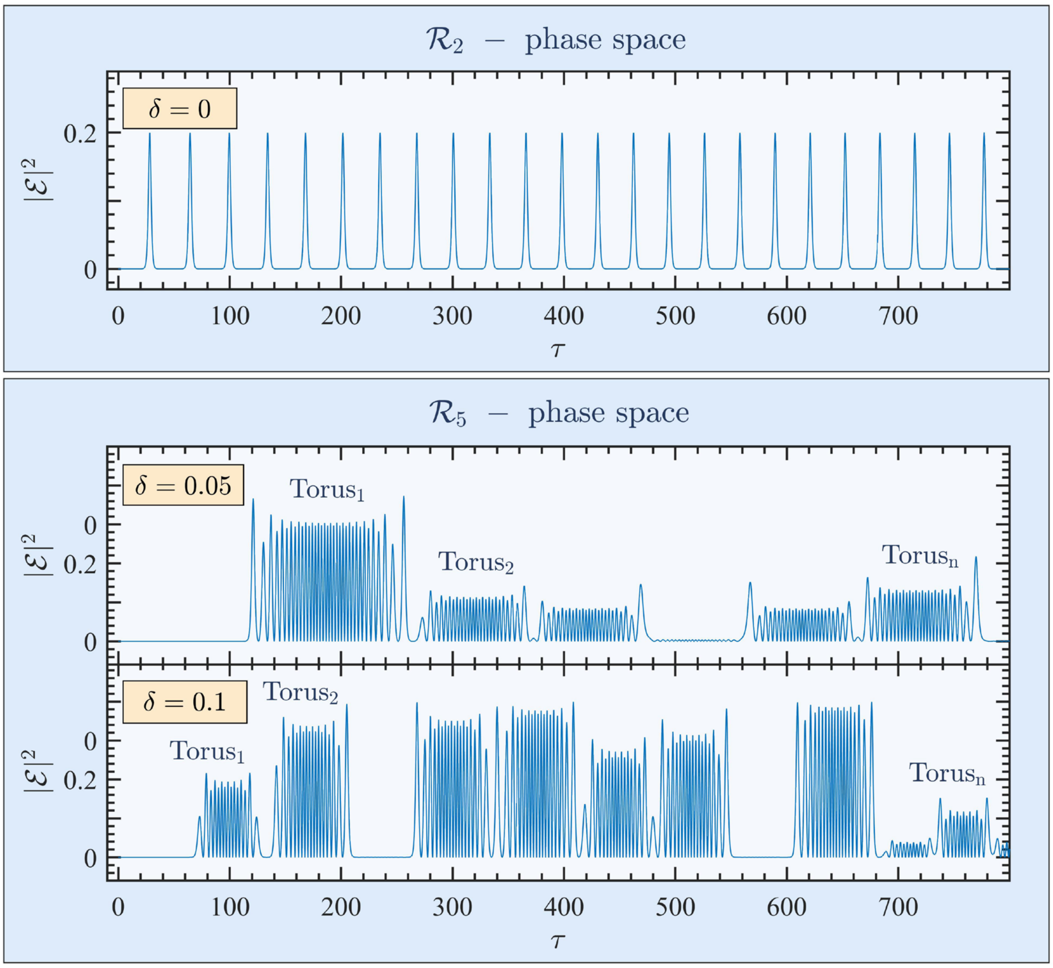

Fig. 2. Dynamics of SR without inversion of different values of the doublet splitting ω_{21}

Conclusions

For multilevel systems, in particular, for systems with the Λ-scheme of operating transitions, we showed that, at any population of the upper level, even without the inversion population on the whole, it is possible to initiate the generation of an SR pulse. The analysis of the new collective basis of the doublet state resulted in the development of conservation laws, which made it possible to considerably reduce the dimension of the phase space of the examined model ($R^{11} \rightarrow R^5 \rightarrow R^2$) and to realise conversion of the model from the complex to the real domain. We show that the system is marked by hyperbolic chaos.

Conflict of Interest

The authors declare that there is no conflict of interest, either existing or potential.

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