Check for updates

Theoretical physics. Cosmology

UDC 524.8
EDN PYVSES
https://www.doi.org/10.33910/2687-153X-2022-3-3-117-121

The forces outside the static limit in the rotating frame

A. A. Grib✉[1], V. D. Vertogradov[1], I. E. Fedorov[1]

[1] Herzen State Pedagogical University of Russia, 48 Moika Emb., Saint Petersburg 191186, Russia

Authors
Andrey A. Grib, ORCID: 0000-0002-6389-991X, e-mail: andrei_grib@mail.ru
Vitalii D. Vertogradov, ORCID: 0000-0002-5096-7696, e-mail: vdvertogradov@gmail.com
Ivan E. Fedorov, e-mail: fedivan97@gmail.com

For citation: Grib, A. A., Vertogradov, V. D., Fedorov, I. E. (2022) The forces outside the static limit in the rotating frame. *Physics of Complex Systems*, 3 (3), 117–121. https://www.doi.org/10.33910/2687-153X-2022-3-3-117-121 EDN PYVSES

Received 15 March 2022; reviewed 25 March 2022; accepted 6 April 2022.

Funding: This study was supported by grant No. 22-22-00112 RSF. It was also part of the SAO RAS state assignment Fundamental Scientific Research.

Abstract. This paper focuses on the rotating frame of the Minkowski spacetime to describe the inertial forces outside the static limit. We consider the inertial forces inside the static limit to find the classical analogue. Then, we find out the expressions for these forces outside the static limit where we cannot consider the limit $\frac{v}{c} \to 0$. We show that, generaly, if the angular velocity of the object Ω is equal to the angular momentum ω, then the acceleration is equal to zero. When $\omega \neq \Omega$, we show that $\omega - \Omega$ is a decreasing function of r.

***Keywords*:** rotating frame, static limit, inertial force, geodesic, Minkowski spacetime

Introduction

The focus on rotating black holes led the authors of (Grib, Pavlov 2019) to conclude that ergosphere, geodesics with negative energies, and the Penrose effect (Grib, Pavlov 2011; Grib et al. 2014; Vertogradov 2015) are typical not only for black holes but also for the Minkowski spacetime in rotating coordinates. It was shown that the role of the static limit in the theory of rotating black holes is taken by the distance from the origin of rotating coordinates to the value of the radius where the linear velocity of rotation becomes equal to the speed of light. The region beyond this value plays the role of ergosphere. In this region all bodies must be moving. This corresponds to use of the rotating rods and presents the main difference from observations inside the limit where all bodies can be at rest. In this paper we consider the geodesic lines both inside and outside the static limit in rotating coordinates. This is done to initiate the discussion of the inertial forces appearing in these coordinates in the Minkowski spacetime. These forces inside the limit are well-known centrifugal and Coriolis forces. The analogues of these forces for the region outside the static limit are presented in this paper.

In the paper (Grib, Pavlov 2019) authors showed that the analogue of the Penrose effect outside the static limit can lead to observational effects inside this static limit. The study reported in this paper is the investigation of a new phenomenon—the inertial forces outside the static limit and their difference from those inside the static limit. It is necessary to take into account the relativistic corrections when one considers the movement of the spacecrafts in this region. This can be achieved by using the rotating frame and observing the signals from the Earth.

In this paper Latin indices take values 0, 1, 2, 3 and Greek indices take values 1, 2, 3.

Inside the static limit

In this paper we consider the flat Minkowski spacetime in cylindrical coordinates {ct, r, φ′, z}:

$$ds^2 = c^2dt^2 - dr^2 - r^2d\varphi'^2 - dz^2 \ . \tag{1}$$

If we consider the rotating coordinates in the Minkowski spacetime

$$\varphi' = \varphi - \omega t \ , \tag{2}$$

then we obtain the following line element:

$$ds^2 = c^2dt^2 - r^2(d\varphi - \omega dt)^2 - dr^2 - dz^2 \ . \tag{3}$$

Here ω is angular velocity.

One can note that the metric component g_{00} changes its sign at $r = r_{sl} = \frac{c}{\omega}$. However, the line element (3) might still be timelike due to the positivity of the off-diagonal term $g_{02}dtd\phi = \omega r^2dtd\phi$. If we compare the metric (3) with the Kerr metric, then the hypersurface $r = r_{sl}$ plays a role of the static limit in the Kerr black hole.

The Killing vector outside the static limit has a different form than inside the static limit i. e. $\frac{d}{dt} + \omega\frac{d}{d\varphi}$. Due to this fact, we have another expression for the proper time where the off-diagonal term is not equal to zero. Thus, the proper time will be imaginary for a body at rest. In this region and the geodesic equations will lead to new phenomenon—the inertial forces.

If we consider the observer on the Earth (motionless with everything rotating around it), then the object which is situated at $r > r_{sl}$ cannot be static for the observer. It must also rotate otherwise the line element (3) will be spacelike. Also, in this case the proper time will be imaginary: if we consider the connection between the proper τ and coordinate t times when $r = const.$, $\phi = const.$, $z = const.$, then we obtain:

$$d\tau = \sqrt{g_{00}}dt \ , \tag{4}$$

and we know that $g_{00} < 0$ at $r > r_{sl}$.

So, if we want to calculate inertial forces for the objects inside the static limit using (4), we have to consider only the region $0 \le r \le r_{sl}$. In this region $g_{00} < 0$. From the Newtonian mechanics we know that acceleration is proportional to force. So, to calculate forces one should use the right hand-side of the geodesic equation:

$$\frac{du^i}{d\tau} = -\Gamma^i_{kl}u^ku^l \ , \tag{5}$$

where Γ^i_{kl} is the Christoffel symbol.

One should note that acceleration is the time derivative of the 4-velocity u^i. From differential geometry we know that the derivative does not obey the tensor transformation law. So, when we define force we should use the covariant derivative instead of the usual one. We want to calculate the three-force, which requires the three covariant derivative instead of (5). Thus, we get (Landau, Lifshitz 1980):

$$f^\alpha = u^\alpha_{;\beta}u^\beta = \frac{du^\alpha}{d\tau} + \gamma^\alpha_{\beta\delta}u^\beta u^\delta \ , \tag{6}$$

where $\gamma^\alpha_{\beta\delta}$ is the purely three-Christoffel symbol. We can note (and show below) that the terms in the right hand-side (5) which are proportional to u^0 correspond to the centrifugal force, while those which

 https://www.doi.org/10.33910/2687-153X-2022-3-3-117-

are proportional to $u^0 u^\alpha$ correspond to the Coriolis force and those which are proportional to $u^\alpha u^\beta$ are part of the three covariant derivative and are not forces at all.

We need the Christoffel symbol of the metric (3) to calculate the geodesic equation (5). Using the well-know formula:

$$\Gamma^i_{kl} = \frac{1}{2} g^{ij}\left(g_{kj,l} + g_{jl,k} - g_{kl,j}\right), \tag{7}$$

one can obtain non-vanishing component of the Christoffel symbols:

$$\begin{aligned} \Gamma^1_{00} &= -\frac{\omega^2 r}{c^2}, \\ \Gamma^1_{02} &= \frac{\omega r}{c}, \\ \Gamma^1_{22} &= -r, \\ \Gamma^2_{12} &= \frac{1}{r}, \\ \Gamma^2_{01} &= -\frac{\omega}{cr}. \end{aligned} \tag{8}$$

To calculate the inertial forces in the region $0 \le r \le r_{sl}$, we can use another well-known formula (Landau, Lifshitz 1980):

$$F_\alpha = \frac{mc^2}{\sqrt{1-\frac{v^2}{c^2}}}\left[-\frac{d}{dx^\alpha}\left(\ln\sqrt{\left(1-\frac{\omega^2 r^2}{c^2}\right)}\right) + \sqrt{1-\frac{\omega^2 r^2}{c^2}}\left(\frac{d\xi_\beta}{dx^\alpha} - \frac{d\xi_\alpha}{dx^\beta}\right)\frac{v^\beta}{c}\right], \tag{9}$$

where

$$\begin{aligned} \xi_\alpha &= \frac{g_{0\alpha}}{1-\frac{\omega^2 r^2}{c^2}}, \\ v^\alpha &= \frac{c\,dx^\alpha}{\sqrt{1-\frac{\omega^2 r^2}{c^2}}\left(c\,dt - \xi_\beta dx^\beta\right)}. \end{aligned} \tag{10}$$

Note that the first term in the square brackets corresponds to the centrifugal force and the second term depends on the velocity linearly and corresponds to the Coriolis force.

Substituting (10) into (9) we obtain:

$$F_r = \left[\frac{mc^2}{\sqrt{1-\frac{v^2}{c^2}}}\left(\frac{2\omega^2 r}{c^2 - \omega^2 r^2}\right)\right]_{centr} + \left[\frac{2\omega r}{\left(1-\frac{\omega^2 r^2}{c^2}\right)^2}\frac{d\varphi}{c\,dt - \xi_2 d\varphi}\right]_{cor}, \tag{11}$$

$$F_\varphi = \left[\frac{2\omega r}{\left(1-\frac{\omega^2 r^2}{c^2}\right)^2}\frac{dr}{c\,dt - \xi_2 d\varphi}\right]_{cor}. \tag{12}$$

Let us find out where the static limit r_{sl} is situated in the Solar system. For this purpose let us consider two cases:

1. The static observer is on the surface of the Earth and everything is rotating around it. In this case we should solve the following algebraic equation:

$$1-\frac{\omega^2 r^2}{c^2}=g_{00}=0\ . \tag{13}$$

The period is equal to 24 h and the static limit is situated between the Uranus and the Neptune.

2. Everything is rotating around the Sun and the observer is on the Earth. In this case the period is equal to 1 year and $r_{sl} = 1.5\times 10^{15} m$. This static limit is situated inside the Oort's clouds.

General case

Here and in what follows we consider the system in units c = 1.

In the general case the proper time τ has the following form:

$$d\tau^2=\left(1-\omega^2 r^2\right)dt^2+2\omega r^2 dt d\varphi-dr^2-dz^2-r^2 d\varphi^2\ . \tag{14}$$

One can see that the square of proper time (14) might be positive in the region $r_{sl}=\frac{1}{\omega}<r<+\infty$ because of the off-diagonal term $+2\omega r^2 dt d\varphi$. Hence we state that we can consider the region $r_{sl} < r < +\infty$ only in the case when the observer has non-vanishing angular velocity $\Omega=\frac{d\varphi}{dt}$. However, we assume that $r = const.$ and $r = const.$ for the observer and the connection between the proper and coordinate times is the following:

$$d\tau=dt\sqrt{1-r^2(\Omega-\omega)^2}\ . \tag{15}$$

To obtain force expression we should write down geodesic equations which, in the general case, are given by:

$$\begin{gathered}\frac{d^2 r}{d\tau^2}=\omega^2 r\left(\frac{dt}{d\tau}\right)^2-2\omega r\frac{dt}{d\tau}\frac{d\varphi}{d\tau}+r\left(\frac{d\varphi}{d\tau}\right)^2\ ,\\ \frac{d^2\varphi}{d\tau^2}=-\frac{2}{r}\frac{dr}{d\tau}\frac{d\varphi}{d\tau}+2\frac{\omega}{r}\frac{dt}{d\tau}\frac{dr}{d\tau}\ ,\end{gathered} \tag{16}$$

If we followed the definition of force from the previous section, then we would see that forces have finite values on the static limit but they are extremely large. No one sees these large values so we should redefine forces.

In the previous section we defined forces as three covariant derivative of the momentum and did not count some terms in the right-hand side of the equation (16) because they were part of the three covariant derivative. However, in the general case, outside the static limit, the Killing vector $\frac{d}{dt}$ is spacelike and as the result is unphysical. In the region $r > r_{sl}$ the observer, like in the ergoregion of a rotating black hole, has to move along both t and φ coordinates. As the result of such movement all the Christoffel symbols proportiaonal to $(u^0)^2$, u^0u^φ and $(u^\varphi)^2$ are inertial forces. So substituting (15) into (16) one obtains:

$$\begin{gathered}\frac{d^2 r}{d\tau^2}=\frac{r}{1-r^2(\omega-\Omega)^2}\left(\Omega-\omega\right)^2\ ,\\ \frac{d^2\varphi}{d\tau^2}=\frac{2v_r}{r-r^3(\omega-\Omega)^2}\left(\omega-\Omega\right),\\ v_r=\frac{dr}{dt}\ .\end{gathered} \tag{17}$$

From (17) one can see that the object has non-vanishing acceleration only if $\omega-\Omega\neq 0$. The object does not have any acceleration if $\Omega=\omega$ because in this case it is equal to zero. If $\omega\neq\Omega$, then the acceleration is a growing function of r and $r=\frac{1}{\omega-\Omega}$ diverges at the surface. This situation is the same like in the

 https://www.doi.org/10.33910/2687-153X-2022-3-3-117-

previous section with the angular velocity ω being replaced by $\omega - \Omega$. If the acceleration of an object is a constant W, then the $\Omega = \frac{d\varphi}{dt}$ is the decreasing function of r, i. e.:

$$\Omega = \omega - \sqrt{\frac{W}{r + Wr^2}} \quad . \tag{18}$$

As is seen, with the proper time (15) we should redefine inertial forces. It leads us to the fact that if $\Omega = \omega$, then the acceleration is absent. However, if the body has constant acceleration, it means that $\Omega \neq \omega$ but $\omega - \Omega$ tends to zero with growing r. Also, one should notice that Ω must always be positive otherwise all metric components become negative outside the static limit and the line element (16) would be spacelike in this region. However, in the region $0 \leq r \leq r_{sl}$ we can consider negative values of ω.

Conclusion

In this paper we have considered forces inside and outside the static limit in the Minkowski spacetime for rotating coordinates. This frame has only two types of inertial forces, i. e., centrifugal and the Coriolis forces. In the case of the static bodies one can consider forces only up to the radius $r = r_{sl}$ because in this case the proper time and velocity is imaginary outside the static limit. Thus, to consider the inertial forces in the region $r_{sl} \leq r \leq \infty$ one should consider the general proper time with non-vanishing angular velocity $\Omega = \frac{d\varphi}{dt}$. We have found out that with the proper time (15), the acceleration tends to zero if $\Omega \to \omega$. It should be also noted that if $\omega \neq \Omega$, then we have another surface $r = \frac{1}{\omega - \Omega}$. This surface shows that the line element outside it (3) is spacelike. That is, the case $\omega \neq \Omega$ is an analogue of the static observer with the angular momentum of spacetime $\omega - \Omega$. It is worth mentioning that all forces which are considered in this article are fictitious ones.

Conflict of Interest

The authors declare that there is no conflict of interest, either existing or potential.

Author Contributions

The authors have made an equal contribution to the preparation of the text.

References

Grib, A. A., Pavlov, Y. V. (2011) Particles with negative energies in black holes. *International Journal of Modern Physics D*, 20 (05), 675–684. https://doi.org/10.1142/S0218271811019013 (In English)

Grib, A. A., Pavlov, Y. V., Vertogradov, V. D. (2014) Geodesics with negative energy in the ergosphere of rotating black holes. *Modern Physics Letters A*, 29 (20), article 1450110. https://doi.org/10.1142/S0217732314501107 (In English)

Grib, A. A., Pavlov, Yu. V. (2019) Static limit and Penrose effect in rotating reference frames. *Theoretical and Mathematical Physics*, 200 (2), 1117–1125. https://doi.org/10.1134/S004057791908004X (In English)

Landau, L. D., Lifshitz, E. M. (1980) *Course of theoretical physics series. Vol. 2. The classical theory of fields*. 4th ed. Oxford: Butterworth-Heinemann Publ., 444 p. (In English)

Vertogradov, V. D. (2015) Geodesics for particles with negative energy in Kerr's metric. *Gravitation and Cosmology*, 21 (2), 171–174. https://doi.org/10.1134/S0202289315020115 (In English)